Foundations of Abstract Interpretation

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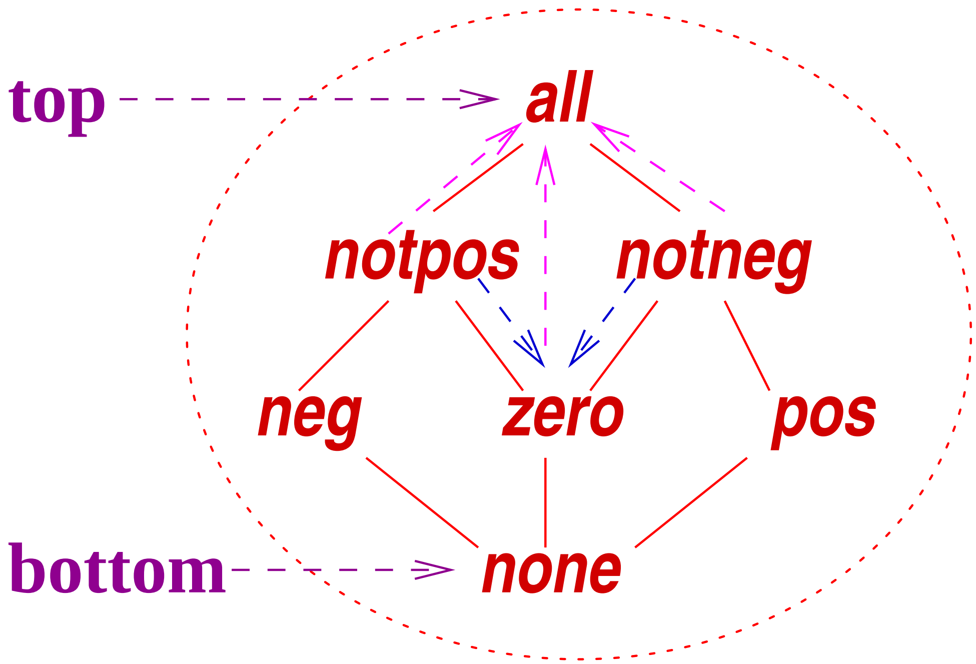
Outline

1. Lattices and continuous functions
2. Galois connections, closures, and Moore families
3. Soundness and completeness of operations on abstract data
4. Soundness and completeness of execution trace computation

Data sets are *complete lattices*

A complete lattice is a partially ordered set, with unique minimal and maximal elements, and with greatest-lower-bound and least-upper-bound operations:



$$\sqcap\{notpos, notneg\} = zero$$

$$\sqcup\{zero, notpos, notneg\} = all$$

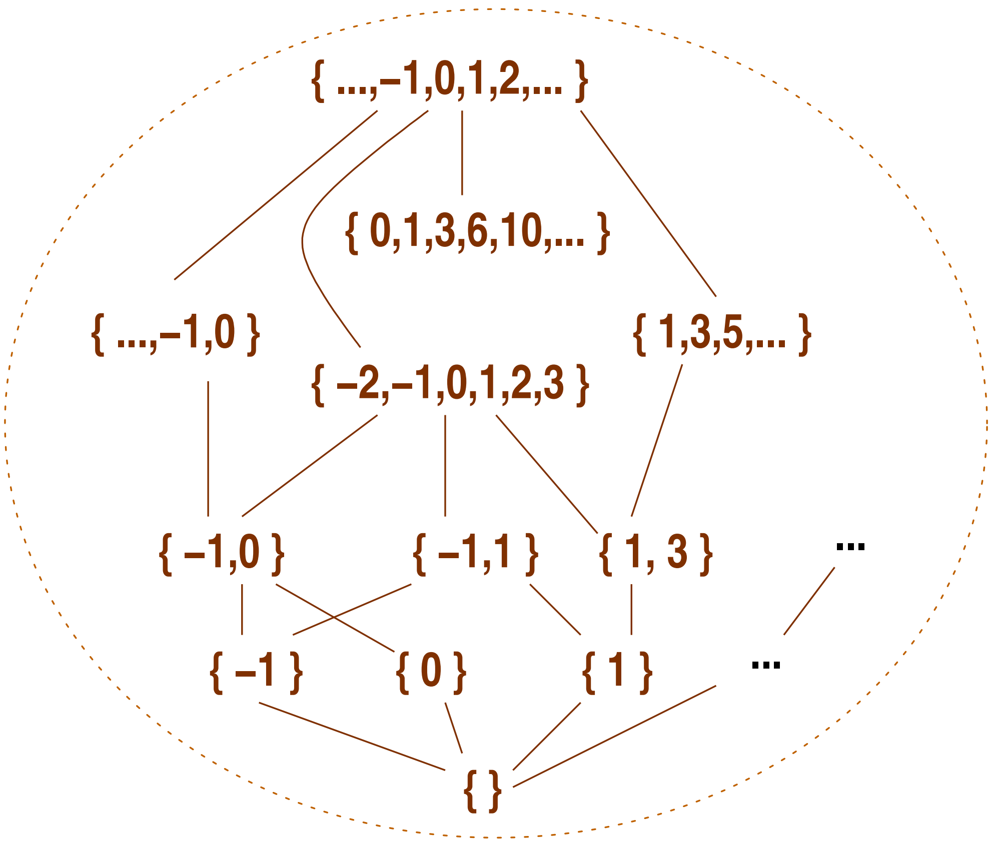
$$\sqcap\{\ \} = all$$

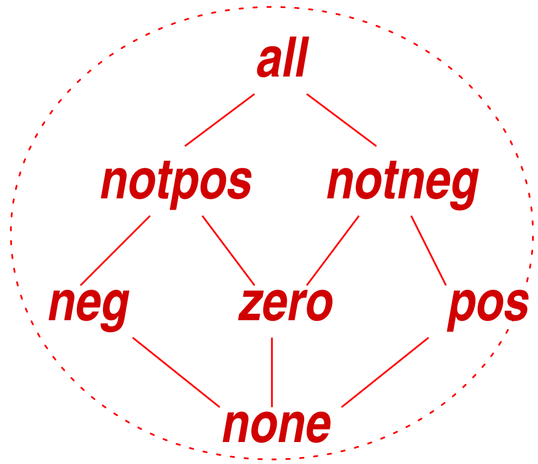
$$\sqcup\{\ \} = none$$

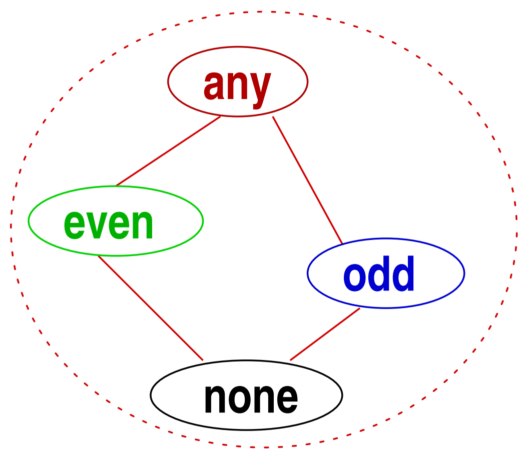
Here is a more precise definition: A *complete lattice*, $\mathcal{L} = \langle \mathrm{D}, \sqsubseteq, \perp, \top, \sqcup, \sqcap \rangle$, consists of

- a set, D, and a partial ordering, $\sqsubseteq$, on D
- a smallest element, $\perp$ (such that $\perp \sqsubseteq \mathrm{d}$, for all $\mathrm{d} \in \mathrm{D}$) and a greatest element, $\top$ (such that $\mathrm{d} \sqsubseteq \top$, for all $\mathrm{d} \in \mathrm{D}$)
- a *least upper bound* operation, $\sqcup$, such that, for all $\mathrm{S} \subseteq \mathrm{D}$, $\mathrm{d} \sqsubseteq \sqcup\mathrm{S}$, for all $\mathrm{d} \in \mathrm{S}$, and for all other upper bounds, $\mathrm{c} \in \mathrm{D}$, such that $\mathrm{d} \sqsubseteq \mathrm{c}$, for all $\mathrm{d} \in \mathrm{S}$, we have that $\sqcup\mathrm{S} \sqsubseteq \mathrm{c}$
- a *greatest lower bound* operation, $\sqcap$, defined dually to the above: $\sqcap\mathrm{S} \sqsubseteq \mathrm{d}$, for all $\mathrm{d} \in \mathrm{S}$, and when $\mathrm{c} \sqsubseteq \mathrm{d}$, for all $\mathrm{d} \in \mathrm{S}$, we have that $\mathrm{c} \sqsubseteq \sqcap\mathrm{S}$

The first example is the complete lattice, $\langle \wp(\mathrm{Int}), \subseteq, \{\,\}, \mathrm{Int}, \bigcup, \bigcap \rangle$; the next two are abstractions of it:







Monotonic and chain-continuous functions

Given complete lattices, $\mathcal{A}$ and $\mathcal{B}$, we say that a function, $f : A \to B$, is *monotonic* iff

$$\text{for all } a, a' \in A, a \sqsubseteq_A a' \text{implies } f(a) \sqsubseteq_B f(a')$$

A monotonic function preserves the "precision of information" in its argument.

Say that we have an ω-*chain*, $a_0 \sqsubseteq_A a_1 \sqsubseteq_A ... \sqsubseteq_A a_i \sqsubseteq_A a_{i+1} \sqsubseteq_A ...$.

A function, $f : A \to B$, is ω-*continuous* iff

$$\bigsqcup_{i \geq 0} f(a_i) = f(\bigsqcup_{i \geq 0} a_i)$$

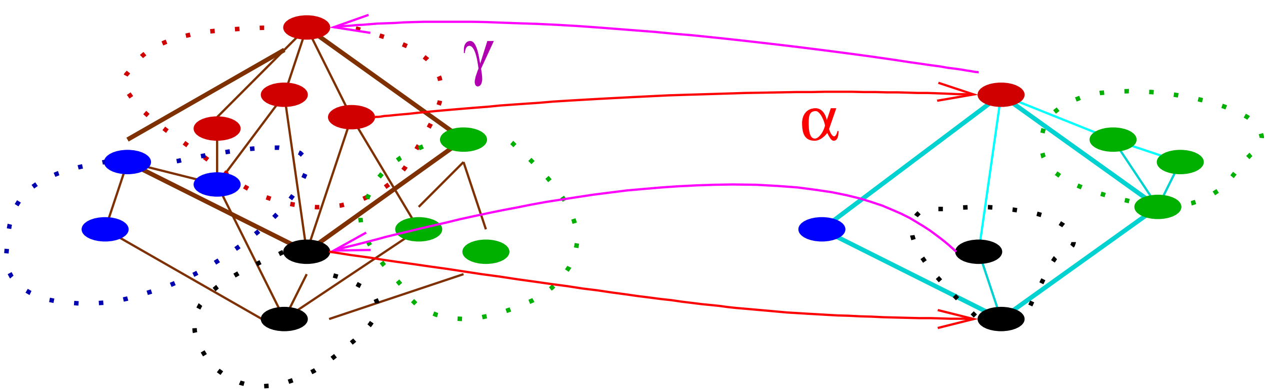
An ω-continuous function preserves the "limit of information" in a chain. Conventional computation employs monotonic and ω-continuous functions, so it is no restriction to use only them.

Galois connections

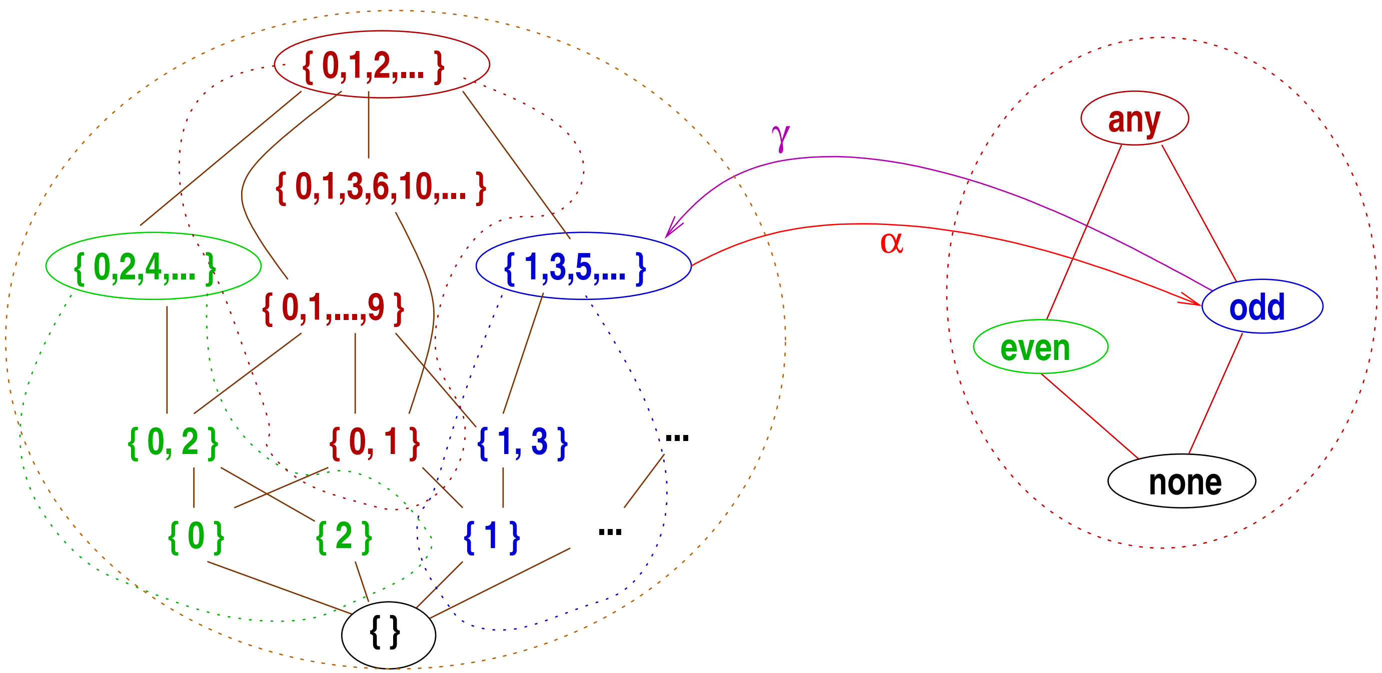
Given a complete lattice of "concrete" (execution) data, $\mathcal{C}$, and a simpler complete lattice of "abstract" data, $\mathcal{A}$, we relate the two by $\alpha : C \to A$ that will act like a *homomorphism* when we study the operations on C.

It will be useful that α have an "inverse,", γ:

Definition: For complete lattices $\mathcal{C}$ and $\mathcal{A}$, and monotonic functions, $\alpha : C \to A$, $\gamma : A \to C$, the pair, $\langle \alpha, \gamma \rangle$ form a *Galois connection*, written $\mathcal{C}\langle \alpha, \gamma \rangle\mathcal{A}$, iff $c \sqsubseteq_C \gamma \circ \alpha(c)$ and $\alpha \circ \gamma(a) \sqsubseteq_A a$.



The maps α and γ are inverse maps on each other's image:



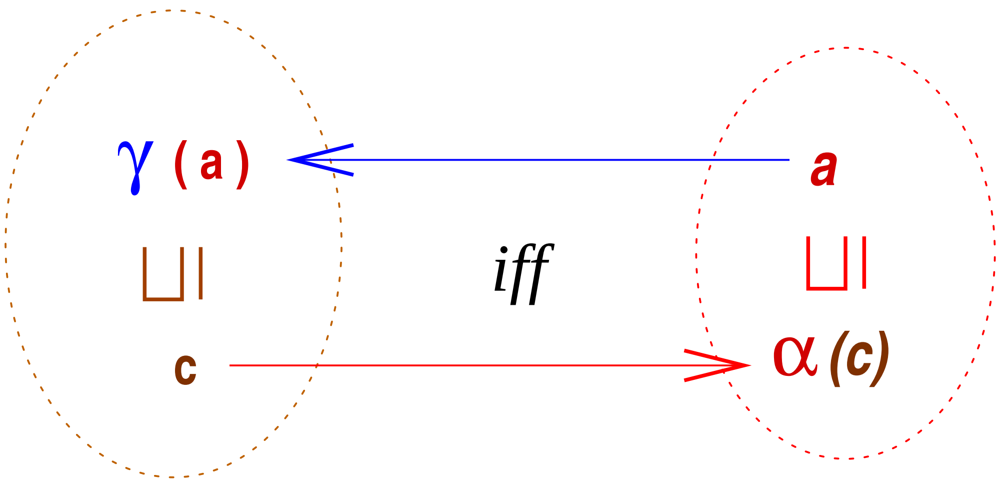
That is, for all $c \in \gamma[A], c = \gamma \circ \alpha(c)$; for all $a \in \alpha[C], a = \alpha \circ \gamma(a)$.

α is ω-continuous (and even preserves $\sqcup$ for arbitrary sets in C); γ preserves $\sqcap$ for arbitrary sets in A. Each map uniquely defines the other:

$$\gamma(a) = \sqcup\{c \mid \alpha(c) \sqsubseteq_A a\} \quad \text{and} \quad \alpha(c) = \sqcap\{a \mid c \sqsubseteq_C \gamma(a)\}$$

The previous fact suggests this alternative characterization of Galois connection:

Proposition: For complete lattices $\mathcal{C}$ and $\mathcal{A}$, the pair, $\langle \alpha : C \to A, \gamma : A \to C \rangle$, is a Galois connection when, for all $c \in C$ and $a \in A$, $c \sqsubseteq_C \gamma(a)$ iff $\alpha(c) \sqsubseteq_A a$.



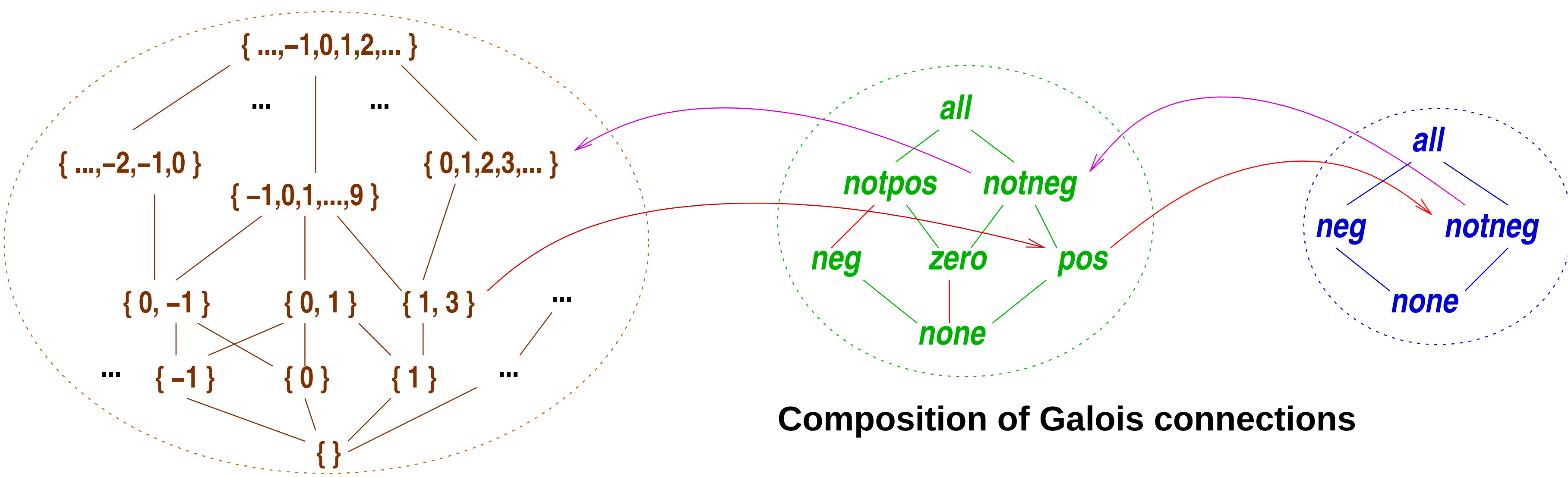
From this definition, we can prove that both α and γ are monotonic, that $c \sqsubseteq_C \gamma \circ \alpha(c)$, and that $\alpha \circ \gamma(a) \sqsubseteq_A a$.

Galois connections are closed under composition, product, and so on:

If $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{D}$ and $\mathcal{D}\langle\alpha',\gamma'\rangle\mathcal{E}$ are Galois connections, **then** so is $\mathcal{C}\langle\alpha'\circ\alpha,\gamma\circ\gamma'\rangle\mathcal{E}$

If $\mathcal{C}_i\langle\alpha_i,\gamma_i\rangle\mathcal{D}_i$ is a Galois connection, for all $i\in I$, **then** so is $\Pi_{i\in I}\mathcal{C}_i\langle\Pi_{i\in I}\alpha_i,\Pi_{i\in I}\gamma_i\rangle\Pi_{i\in I}\mathcal{D}_i$.

If $\mathcal{C}\langle\alpha_C,\gamma_C\rangle\mathcal{C}'$ and $\mathcal{D}\langle\alpha_D,\gamma_D\rangle\mathcal{D}'$ are Galois connections, **then** so is $\mathcal{C}\to\mathcal{D}\langle(\lambda f.\alpha_D\circ f\circ\gamma_C),(\lambda f'.\gamma_D\circ f'\circ\alpha_C)\rangle\mathcal{C}'\to\mathcal{D}'$.



Composition of Galois connections

Why do we require the elaborate structure of a Galois connection?

1. If we are certain about the precise definition of $\gamma : A \longrightarrow C$, we can mechanically synthesize the its adjoint, $\alpha(c) = \sqcap\{a | c \sqsubseteq_C \gamma(a)\}$.

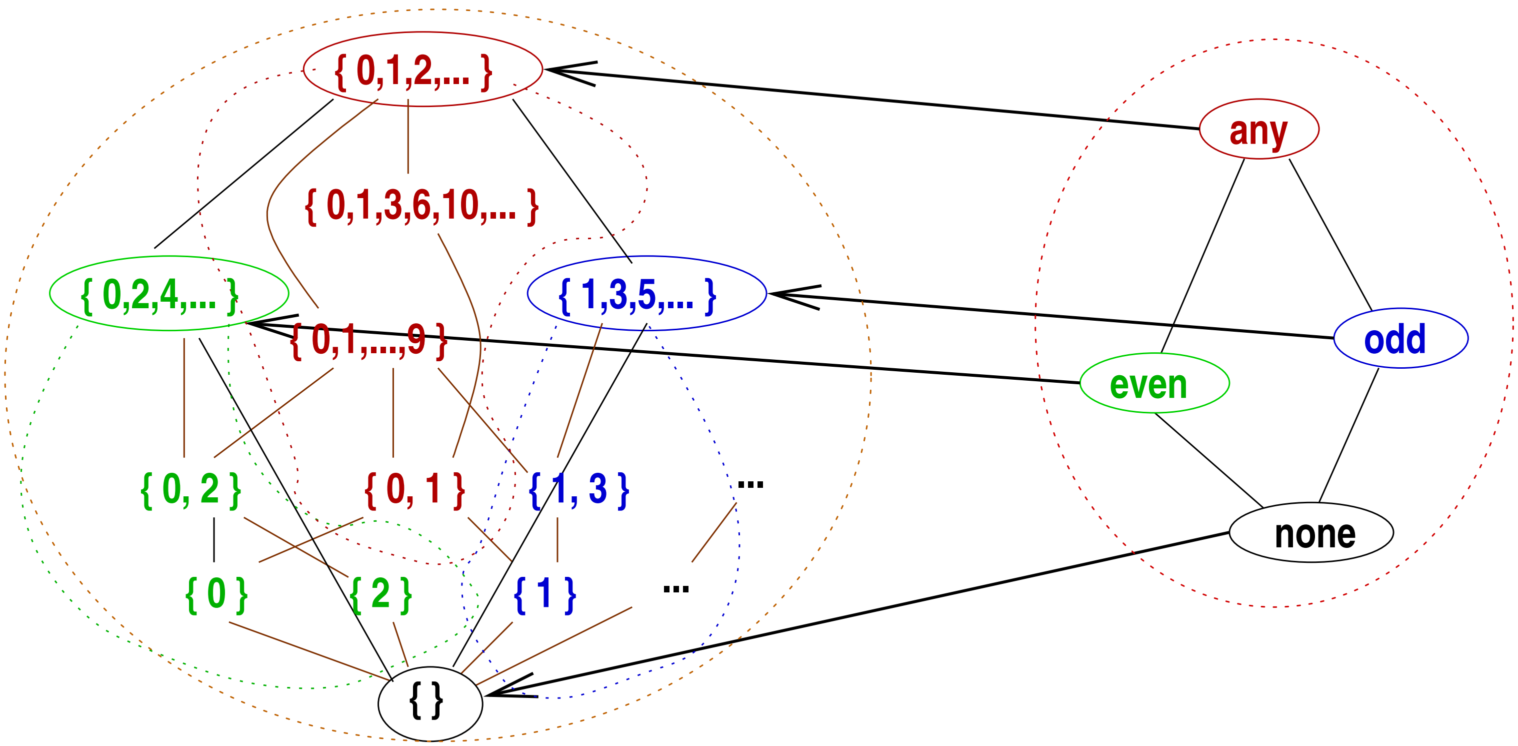
   (Or, dually, if we are certain about α, we can synthesize γ as $\gamma(a) = \sqcup\{c | \alpha(c) \sqsubseteq_A a\}$.)

2. We obtain many mathematical properties about α, expressed in terms of its adjoint, γ (and vice versa).

3. Since we intend to use $\alpha : C \longrightarrow A$ as a “homomorphism” from C to A, we can use α and its adjoint γ to synthesize abstract operations: For each $f : C \longrightarrow C$, we can synthesize $f^{\#} : A \longrightarrow A$, such that α is a “homomorphism” with respect to f and $f^{\#}$. (We will see that $f^{\#} = \alpha \circ f \circ \gamma$.)

Closure maps

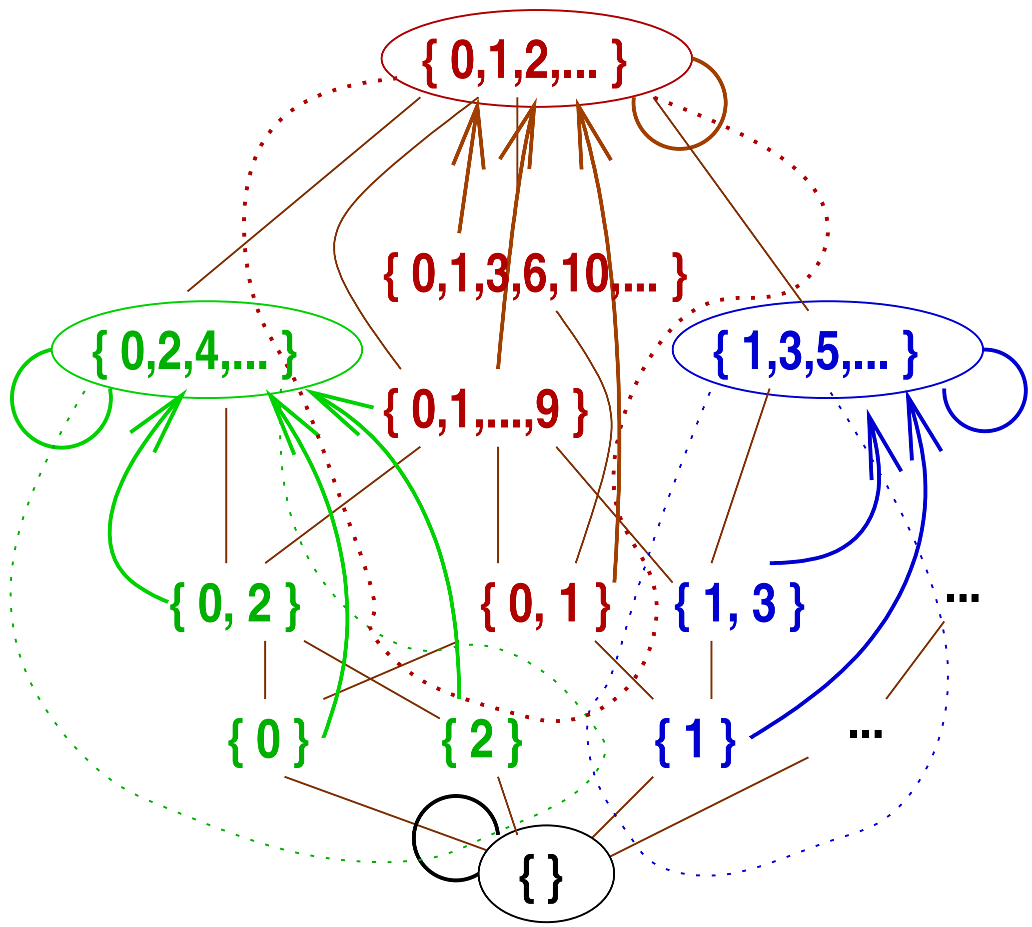
For $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{A}$, it is common that α is onto. This means $\mathcal{A}$ embeds into $\mathcal{C}$ as a sublattice:



A's elements are mere "tokens" that name distinguished sets in C.

Definition: $\rho : \mathcal{C} \to \mathcal{C}$ is a *closure map* if it is **(i)***monotonic*; **(ii)***extensive:* $c \sqsubseteq_C \rho(c)$, for all $c \in C$; **(iii)***idempotent:* $\rho \circ \rho = \rho$.

A closure map defines the embedding:



$\rho\{0,2\} = \{0,2,4,\ldots\}$

$\rho\{0,2,4,\ldots\} = \{0,2,4,\ldots\}$

$\rho\{0,1,\ldots,9\} = \{0,1,2,\ldots\}$

Every Galois connection, $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{A}$, defines a closure map, $\gamma \circ \alpha$.

Every closure map, $\rho : C \rightarrow C$, defines the Galois connection, $\mathcal{C}\langle\rho, \mathrm{id}\rangle\rho[C]$.

Moore families

Given $\mathcal{C}$, can we define a closure map on it by choosing some elements of $\mathcal{C}$? The answer is *yes*, if the elements of C we select are closed under greatest-lower-bounds:

Definition: $M \subseteq C$ is a *Moore family* iff for all $S \subseteq M$, $(\sqcap S) \in M$.

We can define a closure map as $\rho(c) = \sqcap\{c' \in M \mid c \sqsubseteq_C c'\}$.

For a closure map, $\rho : C \to C$, its image, $\rho[C]$, is a Moore family.

Given C, we can define an abstract interpretation by selecting some $M \subseteq C$ that is a Moore family!

Closed binary relations

Often a Galois connection uses a powerset for its concrete domain, that is, $\wp(\mathcal{D})\langle\alpha,\gamma\rangle\mathcal{A}$. This format yields a simple characterization:

Given unordered set D and complete lattice A, it is natural to relate the elements in D to those in A by a binary relation, $\mathcal{R} \subseteq D \times A$, such that $(d, a) \in \mathcal{R}$ means "d *has property* a." We write this as $d\ \mathcal{R}\ a$ or as $d \models_{\mathcal{R}} a$.

Example: $D = \mathrm{Int}$, and $A = \{\mathrm{none}, \mathrm{neg}, \mathrm{pos}, \mathrm{zero}, \mathrm{nonneg}, \mathrm{nonpos}, \mathrm{any}\}$.

Then, $2\ \mathcal{R}\ \mathrm{nonneg}$, $2\ \mathcal{R}\ \mathrm{pos}$, and $2\ \mathcal{R}\ \mathrm{any}$. (Or we write, $2 \models_{\mathcal{R}} \mathrm{nonneg}$, $2 \models_{\mathcal{R}} \mathrm{pos}$, and $2 \models_{\mathcal{R}} \mathrm{any}$.)

We immediately define the function, $\gamma : A \to \wp(D)$, as

$$\gamma(a) = \{d \in D \mid d\ \mathcal{R}\ a\}$$

For example, $\gamma(\mathrm{nonneg}) = \{0, 1, 2, ...\}$.

We can check if γ is the upper adjoint of a Galois connection, say, by showing that $\gamma[A]$ defines a Moore family. But we can check for this directly upon $\mathcal{R}$:

Proposition: $\mathcal{R} \subseteq D \times A$ defines a Galois connection between $\wp(D)$ and A iff **(i)** $\mathcal{R}$ is *U-closed*: $c\ \mathcal{R}\ a$ and $a \sqsubseteq_A a'$ imply $c\ \mathcal{R}\ a'$; **(ii)** $\mathcal{R}$ is *G-closed*: $c\ \mathcal{R}\ \sqcap\{a \mid c\ \mathcal{R}\ a\}$.

If $\mathcal{R}$ defines a Galois connection, then we have this crucial property:

- for all $a \in A$ and $C \in \wp(D)$, $C \subseteq \gamma(a)$ iff $\alpha(C) \sqsubseteq_A a$ iff ($c\ \mathcal{R}\ a$, for all $c \in C$).

This is of course the definition of a Galois connection, and in this sense, $\mathcal{R}$ “is” a Galois connection.

A recipe for abstract-domain building

Given an unordered set, D, of concrete data values, we might ask, *"What are the properties about* D *that I wish to calculate? Can I relate these properties,* $a \in A$*, to elements* $d \in \mathrm{D}$ *via a UG-closed binary relation,* $\mathcal{R}_{\mathrm{D}} \subseteq \mathrm{D} \times A$*?"* Given a set, A, and relation, $\mathcal{R}_{\mathrm{D}} \subseteq \mathrm{D} \times A$,

1. Define $\gamma : A \rightarrow \wp(\mathrm{D})$ as $\gamma(a) = \{d \mid d\ \mathcal{R}_{\mathrm{D}}\ a\}$.
2. Define this partial ordering on A: $a \sqsubseteq a'$ iff $\gamma(a) \subseteq \gamma(a')$. (If there are distinct $a, a' \in A$ such that $\gamma(a) = \gamma(a')$, then merge them.) This forces U-closure.
3. Ensure that $\gamma[A]$ is a Moore family by adding greatest-lower-bound elements to A as needed. This forces G-closure.
4. Use the existing machinery to define the Galois connection between $\wp(\mathrm{D})$ and A.

Example: Abstracting the Program State

The concrete storage vector is a product,

$$\mathrm{Store} = \Pi_{i \in \mathrm{Identifier}} \mathrm{Data}$$

and the concrete program state is a $\mathrm{ProgramPoint} \times \mathrm{Store}$ pair.

Example: $p_1, \langle x : 3, y : 4 \rangle$ is a program state.

Say that we have the relation, $\mathcal{R}_{\mathrm{Data}} \subseteq \mathrm{Data} \times \mathrm{AbsData}$, and we have the induced Galois connection, $\wp(\mathrm{Data})\langle \alpha_{\mathrm{Data}}, \gamma_{\mathrm{Data}} \rangle \mathrm{AbsData}$. Now, we can build Galois connections that abstract the store and the state.

A concrete store is related to an abstract store:

$$\langle x_i : v_i \rangle_{i \in \mathrm{Id}} \; \mathcal{R}_{\mathrm{Store}} \; \langle x_i : a_i \rangle_{i \in \mathrm{Id}}, \text{ iff, for all } i \in \mathrm{Id}, v_i \; \mathcal{R}_{\mathrm{Data}} \; a_i$$

Example: $\langle x : 3, y : 4 \rangle \; \mathcal{R}_{\mathrm{Store}} \; \langle x : any, y : even \rangle$.

This produces a Galois connection, $\wp(\mathrm{Store})\langle \alpha_{\mathrm{Store}}, \gamma_{\mathrm{Store}} \rangle \mathrm{AbsStore}$, where $\mathrm{AbsStore} = \Pi_{i \in \mathrm{Identifier}} \mathrm{AbsData}$ and

$$\gamma\langle x_i : a_i\rangle_{i\in \mathrm{Id}} = \{\langle x_i : v_i\rangle_{i \in \mathrm{Id}} \mid v_i \in \gamma_{\mathrm{Data}}(a_i), \text{ for all } i \in \mathrm{Id}\}$$

$$\alpha_{\mathrm{Store}}(S) = \langle \bigsqcup_{s\in S} \alpha(s(i))\rangle_{i\in \mathrm{Id}}$$

For example,

$\gamma_{\mathrm{Store}}\langle x : \mathrm{even}, y : \mathrm{odd}\rangle = \{\langle x : 0, y : 1\rangle, \langle x : 0, y : 3\rangle, \langle x : 2, y : 1\rangle, ...\}$

A program point is abstracted to itself: $p \ \mathcal{R}_{\mathrm{PP}} \ p$, suggesting that the abstract domain of program points might be merely $\mathrm{AbsPP} = \mathrm{ProgramPoint} \cup \{\bot, \top\}$. ($\top$ and $\bot$ are needed to make AbsPP a complete lattice.)

Finally, we can relate a concrete state to an abstract one:

$$p, s \ \mathcal{R}_{\mathrm{State}} \ p', \sigma \text{ iff } p \ \mathcal{R}_{\mathrm{PP}} \ p' \text{ and } s \ \mathcal{R}_{\mathrm{Store}} \ \sigma$$

Hence, $\gamma_{\mathrm{State}}(p_i, \sigma) = \{p_i, s \mid s \in \gamma_{\mathrm{Store}}(\sigma)\}$.

Concrete and abstract operations

Now that we know how to model $c \in C$ by $\alpha(c) \in A$, we must model concrete computation steps, $f : C \to C$, by abstract computation steps, $f^{\#} : A \to A$.

Example: We have concrete domain, Nat, and concrete operation, $succ : \mathrm{Nat} \to \mathrm{Nat}$, defined as $succ(n) = n + 1$.

We have abstract domain, $Parity$, and abstract operation, $succ^{\#} : Parity \to Parity$, defined as

$$succ^{\#}(even) = odd, \quad succ^{\#}(odd) = even$$

$$succ^{\#}(any) = any, \quad succ^{\#}(none) = none$$

$succ^{\#}$ must be consistent (sound) with respect to $succ$:

if $n \ \mathcal{R}_{\mathrm{Nat}} \ a$, then $succ(n) \ \mathcal{R}_{\mathrm{Nat}} \ succ^{\#}(a)$

where $\mathcal{R} \subseteq \mathrm{Nat} \times Parity$ relates numbers to their parities (e.g., $2 \ \mathcal{R}_{\mathrm{Nat}} \ even$, $5 \ \mathcal{R}_{\mathrm{Nat}} \ odd$, etc.).

We want *soundness*: $n \ \mathcal{R}_{Nat} \ a$ implies $succ(n) \ \mathcal{R}_{Nat} \ succ^{\#}(a)$, for all $n \in Nat$ and $a \in Parity$.

Since we have the Galois connection, $\wp(Nat)\langle\alpha,\gamma\rangle Parity$, we know that $\gamma(a) = \{n \mid n \ \mathcal{R}_{Nat} \ a\}$.

So, soundness is stated equivalently as

$$\text{for all } a \in A, \text{ for all } n \in \gamma(a), \ succ(n) \in \gamma(succ^{\#}(a))$$

and this is equivalent to saying,

$$\text{for all } a \in A, \ succ^*(\gamma(a)) \subseteq_{Nat} \gamma(succ^{\#}(a))$$

that is,

$$\text{for all } a \in A, \ (succ^* \circ \gamma)(a) \subseteq_{Nat} (\gamma \circ succ^{\#})(a)$$

where $succ^* : \wp(Nat) \rightarrow \wp(Nat)$ is $succ^*(S) = \{succ(n) \mid n \in S\}$.

This is interesting, because it states a commutative, "semi-homorphism" property....

Definition: For Galois connection, $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{A}$, and functions $f : C \to C$, $f^{\#} : A \to A$, $f^{\#}$ is a *sound approximation* of f iff

$$(\alpha \circ f)(c) \sqsubseteq_A (f^{\#} \circ \alpha)(c), \text{ for all } c \in C$$

iff

$$(f \circ \gamma)(a) \sqsubseteq_C (\gamma \circ f^{\#})(a), \text{ for all } a \in A$$

This slightly abstract presentation exposes that α is a "semi-homomorphism" with respect to f and $f^{\#}$:

$$\begin{tikzcd}
c \arrow[rrr, "\alpha"] \arrow[d, "f"'] & & & \alpha(c) \arrow[d, "f^{\#}"] \\
f(c) \arrow[r, "\alpha"] & \alpha(f(c)) & \sqsubseteq & f^{\#}(\alpha(c))
\end{tikzcd}$$

Example 1: $n\ \mathcal{R}_{\mathrm{Nat}}\ a$ implies $\mathrm{succ}(n)\ \mathcal{R}_{\mathrm{Nat}}\ \mathrm{succ}^{\#}(a)$

Galois connection: $\wp(\mathrm{Nat})\langle\alpha,\gamma\rangle\mathrm{Parity}$

$\mathrm{succ}^* : \wp(\mathrm{Nat}) \to \wp(\mathrm{Nat})$

$\mathrm{succ}^*(S) = \{\mathrm{succ}(n) \mid n \in S\}$

where $\mathrm{succ}(n) = n + 1$

$\mathrm{succ}^{\#} : \mathrm{Parity} \to \mathrm{Parity}$

$\mathrm{succ}^{\#}(\mathrm{even}) = \mathrm{odd}, \quad \mathrm{succ}^{\#}(\mathrm{odd}) = \mathrm{even}$

$\mathrm{succ}^{\#}(\mathrm{any}) = \mathrm{any}, \quad \mathrm{succ}^{\#}(\mathrm{none}) = \mathrm{none}$

We have that $\alpha \circ \mathrm{succ}^* = \mathrm{succ}^{\#} \circ \alpha$:

$$\begin{tikzcd}
\{2,6\} \arrow[rr, "\alpha"] \arrow[d, "\mathbf{succ}\,^*"'] & & \mathbf{even} \arrow[d, "\mathbf{succ}\,^{\#}"] \\
\{3,7\} \arrow[r, "\alpha"] & \mathbf{odd} \arrow[r, phantom, "="] & \mathbf{odd}
\end{tikzcd}$$

Example 2: $n \; \mathcal{R}_{\mathrm{Nat}} \; a$ implies $\mathrm{div2}(n) \; \mathcal{R}_{\mathrm{Nat}} \; \mathrm{div2}^{\#}(a)$

Galois connection: $\wp(\mathrm{Nat})\langle\alpha, \gamma\rangle\mathrm{Parity}$

$$\mathrm{div2}^{*} : \wp(\mathrm{Nat}) \to \wp(\mathrm{Nat})$$

$$\mathrm{div2}^{*}(S) = \{\mathrm{div2}(n) \mid n \in S\}$$

where $\mathrm{div2}(2n+1) = \mathrm{div2}(2n) = n$

$$\mathrm{div2}^{\#} : \mathrm{Parity} \to \mathrm{Parity}$$

$$\mathrm{div2}^{\#}(\mathrm{even}) = \mathrm{div2}^{\#}(\mathrm{odd}) = \mathrm{any}$$

$$\mathrm{div2}^{\#}(\mathrm{any}) = \mathrm{any}, \quad \mathrm{div2}^{\#}(\mathrm{none}) = \mathrm{none}$$

We have that $\alpha \circ \mathrm{div2}^{*} \sqsubseteq_{\mathrm{Parity}} \mathrm{div2}^{\#} \circ \alpha$:

$$\begin{tikzcd}
\{6\} \arrow[rr, "\alpha"] \arrow[d, "\mathbf{div2}^{*}"'] & & \mathbf{even} \arrow[d, "\mathbf{div2}^{\#}"] \\
\{3\} \arrow[r, "\alpha"] & \mathbf{odd} \arrow[r, phantom, "\sqsubseteq"] & \mathbf{any}
\end{tikzcd}$$

Synthesizing $f^\#$ from f

The previous slides show how α acts as a "semi-homomorphism" between f and $f^\#$.

Given the Galois connection, $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{A}$, and operation, $f : C \rightarrow C$, the most precise $f^\#_{best} : A \rightarrow A$ that is sound with respect to f is

$$f^\#_{best} = \alpha \circ f \circ \gamma$$

Proposition: $f^\#$ is sound with respect to f iff $f^\#_{best} \sqsubseteq_{A\rightarrow A} f^\#$.

(Note: $f \sqsubseteq_{A\rightarrow A} g$ iff for all $a \in A$, $f(a) \sqsubseteq_A g(a)$.)

Of course, $f^\#_{best}$ has a *mathematical* definition — not an algorithmic one — $f^\#_{best}$ might not be finitely computable !

Parity example continued:

$$\begin{aligned} succ^\#_{best}(even) &= \alpha \circ succ^*(\gamma\ even) \\ &= \alpha(succ^*\{2n \mid n \geq 0\}) \\ &= \alpha\{2n+1 \mid n \geq 0\} = odd \end{aligned}$$

One more example:

Given $\wp(Nat)\langle\alpha,\gamma\rangle Parity$ and $div2 : Nat \rightarrow Nat$, we have

$$div2^* : \wp(Nat) \rightarrow \wp(Nat)$$
$$div2^*(S) = div2[S] = \{div2(n) \mid n \in S\}$$

Hence, $div2^{\#}_{best} = \alpha \circ div2^* \circ \gamma$. The operation loses precision: $\alpha(div2^*\{4\}) = \alpha\{2\} = even$, but

$$div2^{\#}_{best}(even) = \alpha(div2^*(\gamma(even)))$$
$$= \alpha(div2^*\{0, 2, 4, ...\})$$
$$= \alpha\{1, 2, 3, ...\} = any$$

Nonetheless, this is the best we can do, given the crude structure of the abstract domain, **Parity**.

Completeness

Given $\mathcal{C}\langle\alpha,\gamma\rangle\mathcal{A}$, we state soundness of $f^{\#}$ with respect to f as $\alpha \circ f \sqsubseteq_{A \to A} f^{\#} \circ \alpha$ iff $f \circ \gamma \sqsubseteq_{C \to C} \gamma \circ f^{\#}$

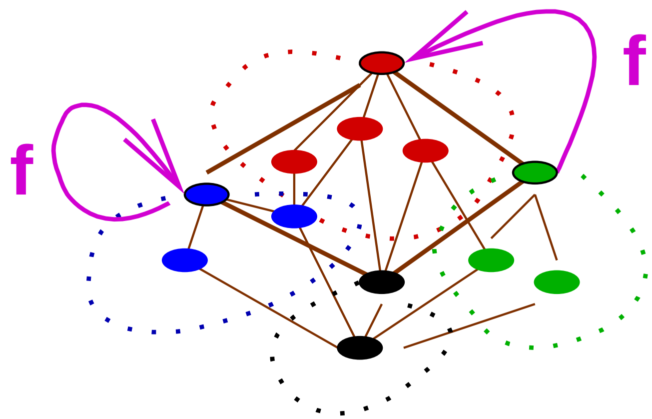
Definition: $f^{\#}$ is *forwards (γ) complete* with respect to f iff $f \circ \gamma =_{C \to C} \gamma \circ f^{\#}$

Definition: $f^{\#}$ is *backwards (α) complete* with respect to f iff $\alpha \circ f =_{A \to A} f^{\#} \circ \alpha$

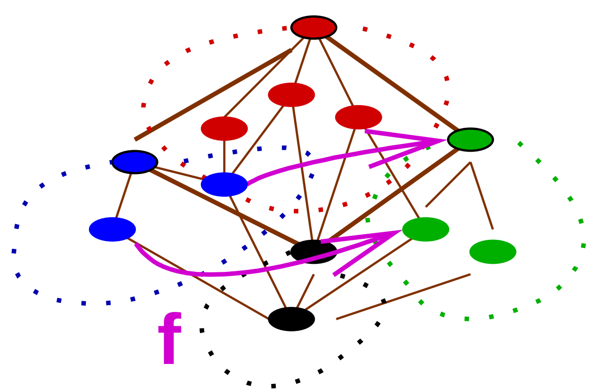
The two completeness notions are not equivalent!

For an $f^{\#}$ to be (forwards or backwards) complete, it must equal $f^{\#}_{\mathrm{best}} = \alpha \circ f \circ \gamma$. Indeed, the structure of the Galois connection and $f : C \to C$ determines whether $f^{\#}_{\mathrm{best}}$ is complete.

Forwards (γ) completeness: $f^{\#}_{best}$ is forwards-complete iff f maps image points of γ to image points of γ — $f(\gamma[A]) \subseteq \gamma[A]$.



Backwards (α) completeness: $f^{\#}_{best}$ is backwards-complete iff f maps all points in the same α-equivalence class to points in the same α-equivalence class — $\alpha(c) = \alpha(c')$ implies $\alpha(f(c)) = \alpha(f(c'))$.



Transfer functions generate computation steps

Each program transition from program point p_i to p_j has an associated *transfer function*, $f_{ij} : C \to C$ (or $f^{\#}_{ij} : \mathcal{A} \to \mathcal{A}$), which describes the associated computation.

This defines a computation step of the form, $p_i, s \to p_j, f_{ij}(s)$.

Example: Assignment $p_0 : \mathtt{x\ =\ x+1;}\ p_1 : \cdots$ has the transfer function, $f_{01}\langle ...\mathtt{x} : n... \rangle = \langle ...\mathtt{x} : n + 1... \rangle$. For example, $p_0, \langle \mathtt{x} : 3 \rangle \to p_1, f_{01}\langle \mathtt{x} : 3 \rangle = p_1, \langle \mathtt{x} : 4 \rangle$.

For modelling multiple transitions in conditional/nondeterministic choice, we attach a transfer function to each possible transition.

Example: For

p_0 : `cases`

    x ≤ y: p1 : y = y - x;
    y ≤ x: p2 : x = x - y;

`end`

For

```
p0 : cases
   x ≤ y:  p1 : y = y - x;
   y ≤ x:  p2 : x = x - y;
end
```

we have these functions:

$$f_{01}(s) = \begin{cases} s \text{ if } s.x \leq s.y \\ \perp \text{ otherwise} \end{cases}$$

$$f_{02}(s) = \begin{cases} s \text{ if } s.y \leq s.x \\ \perp \text{ otherwise} \end{cases}$$

For example, $p_0, \langle x:5, y:3\rangle \rightarrow p_1, \perp$, because $x \not\leq y$, but $p_0, \langle x:5, y:3\rangle \rightarrow p_2, \langle x:5, y:3\rangle$, because $y \leq x$. The transfer functions "filter" the data that arrives at a program point.

We ignore computation steps, $p, s \rightarrow p', \perp$, that produce "no data" ($\perp$).

An *execution trace* is a (possibly infinite) sequence, $p_0, s_0 \rightarrow p_1, s_1 \rightarrow \cdots \rightarrow p_j, s_j \rightarrow \cdots$, such that, for all $i \geq 0$:

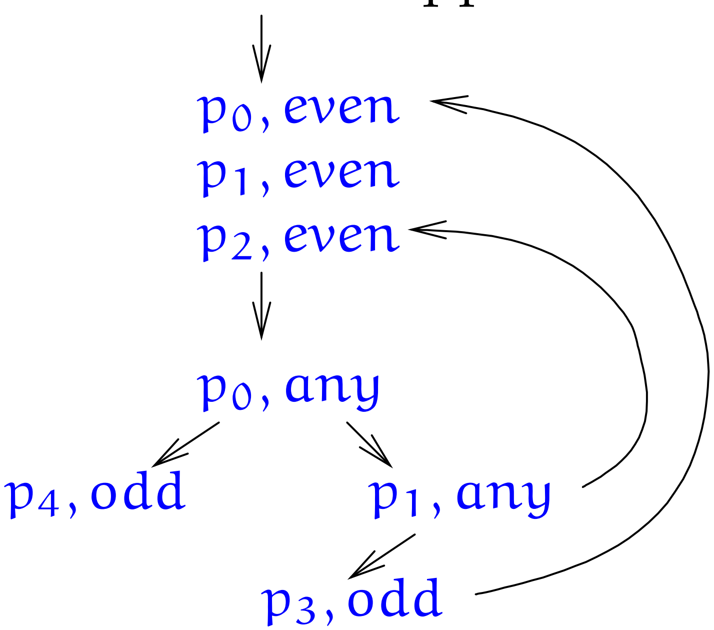
- $p_i, s_i \rightarrow p_{succ(i)}, f_{i,succ(i)}(s_i)$
- no s_i equals $\perp$.

Using the $f^{\#}$s to build sound, abstract trace trees

```
p0 : while (x != 1) {
  p1 : if Even(x)
    p2 : then  x = x div2;
    p3 : else  x = 3*x + 1;
  }
p4 :  exit
```

Note: p_i, v abbreviates $p_i, \langle x : v \rangle$

Abstract overapproximating trace:



Two concrete traces:

$\downarrow$	$\downarrow$
$p_0, 4$	$p_0, 6$
$p_1, 4$	$p_1, 6$
$p_2, 4$	$p_2, 6$
$\downarrow$	$\downarrow$
$p_0, 2$	$p_0, 3$
$p_1, 2$	$p_1, 3$
$p_2, 2$	$p_2, 3$
$\downarrow$	$\downarrow$
$p_0, 1$	$p_0, 10$
$p_4, 1$	$\cdots$
	$\downarrow$
	$p_4, 1$

Each concrete transition is generated by an f_{ij}; each abstract transition is generated by the corresponding $f^{\#}_{ij}$.

Each concrete transition, $p_i, s \rightarrow p_j, f_{ij}(s)$, is reproduced by a corresponding abstract transition, $p_i, a \rightarrow p_j, f^{\#}_{ij}(a)$, where $s \in \gamma(a)$.

For example, $p_2 :$ `x = x div2` is interpreted *concretely* by $f_{20}(2n) = n = f_{20}(2n+1)$ and is interpreted *abstractly* by $f^{\#}_{20}(even) = any = f^{\#}_{20}(odd) = f^{\#}_{20}(any)$.

The traces embedded in the abstract trace tree "cover" (*simulate*) the concrete traces, e.g., this concrete trace,

$$p_0, 4 \rightarrow p_1, 4 \rightarrow p_2, 4 \rightarrow p_0, 2 \rightarrow p_1, 2 \rightarrow p_2, 2 \rightarrow p_0, 1 \rightarrow p_4, 1$$

is simulated by this abstract trace, which is extracted from the abstract computation tree:

$$p_0, even \rightarrow p_1, even \rightarrow p_2, even \rightarrow p_0, any \rightarrow p_1, any \rightarrow p_2, even \rightarrow p_0, any \rightarrow p_4, odd$$

and indeed, *all* concrete traces starting with $p_0, 2n$, $n >= 0$, are simulated by the abstract tree in this manner.

Proof of soundness of trace construction

For $S \in C$ and $a \in A$, say that $S \,\mathcal{R}\, a$ iff $S \sqsubseteq_C \gamma(a)$ iff $\alpha(S) \sqsubseteq_A a$.

Lemma: $\alpha \circ f \sqsubseteq_{A \to A} f^\# \circ \alpha$ **iff** $f \circ \gamma \sqsubseteq_{C \to C} \gamma \circ f^\#$ **iff** $S \,\mathcal{R}\, a$ implies $f(S) \,\mathcal{R}\, f^\#(a)$.

Theorem: For every concrete trace, $(p_i, s_i)_{i \geq 0}$, there exists an abstract trace, $(p_i, a_i)_{i \geq 0}$, such that for all $i \geq 0$, $\{s_i\} \,\mathcal{R}\, a_i$.

Proof: We use the Lemma and induction to assemble this diagram:

$$\begin{array}{lllll}
p_0, s_0 \longrightarrow & p_1, f_0(s_0) = p_1, s_1 & \longrightarrow p_2, f_1(s_1) = p_2, s_2 \longrightarrow & \cdots \longrightarrow p_i, s_i & \longrightarrow \cdots \\
\mathcal{R} \,\vdots & \mathcal{R} \,\vdots & \mathcal{R} \,\vdots & \mathcal{R} \,\vdots & \\
p_0, a_0 \longrightarrow & p_1, f_0^\#(a_0) = p_1, a_1 & \longrightarrow p_2, f_1^\#(a_1) = p_2, a_2 \longrightarrow & \cdots \longrightarrow p_i, a_i & \longrightarrow \cdots
\end{array}$$

(Note: each s_i in the diagram is more precisely stated as $\{s_i\}$, because $C = \wp(\mathrm{Store})$.) Due to imprecision of the $f^\#$s, the abstract trace tree may possess many traces that begin with p_0, a_0, but there is always one trace in the tree that simulates the concrete trace.

When all the operations, $f^{\#}_{ij}$, are complete with respect to the f_{ij}s, the previous result is strengthened:

Say that $S \;\mathcal{R}\; a$ iff $\alpha(S) = a$. (Similarly, say that $S \;\mathcal{R}\; a$ iff $S = \gamma(a)$.)

In both cases, the lemma holds:

Lemma: $\alpha \circ f =_{A \to A} f^{\#} \circ \alpha$ **iff** $S \;\mathcal{R}\; a$ implies $f(S) \;\mathcal{R}\; f^{\#}(a)$.
(Similarly, $f \circ \gamma =_{C \to C} \gamma \circ f^{\#}$ **iff** $S \;\mathcal{R}\; a$ implies $f(S) \;\mathcal{R}\; f^{\#}(a)$.)

Theorem (α-completeness): When $S \;\mathcal{R}\; a$ iff $\alpha(S) = a$, then for every concrete trace, $(p_i, s_i)_{i \geq 0}$, there exists an abstract trace, $(p_i, a_i)_{i \geq 0}$, such that for all $i \geq 0$, $\{s_i\} \;\mathcal{R}\; a_i$.

Theorem (γ-completeness): When $S \;\mathcal{R}\; a$ iff $\gamma(a) = S$, $S \subseteq \mathit{Store}$, then for every trace on *sets of stores*, $(p_i, S_i)_{i \geq 0}$, there exists an abstract trace, $(p_i, a_i)_{i \geq 0}$, such that for all $i \geq 0$, $S_i \;\mathcal{R}\; a_i$.

References

- P. Cousot and R. Cousot. Abstract interpretation: a unifi ed lattice model for static analysis of programs by construction or approximation of fi xpoints. ACM POPL 1977.
- P. Cousot and R. Cousot. Systematic design of program analysis frameworks. ACM POPL, 1979.
- P. Cousot. Slides for invited lecture at VMCAI 2003, New York City. www.di.ens.fr/~cousot
- R. Giacobazzi, F. Ranzato, and F. Scozzari. Making abstract interpretations complete. Journal ACM 47(2) 2000.
- R. Giacobazzi and E. Quinterelli. Incompleteness, counterexamples, and refi nements in abstract model-checking. SAS 2001, Springer LNCS 2126.
- D. Schmidt. Structure-preserving binary relations for program abstraction. In The Essence of Computation: Complexity, Analysis, Transformation. Springer LNCS 2566, 2002.