Behavioural equivalences

Comparing behaviours

- terms and their *meaning*:

a syntactic object	its $\equiv$-equivalence class	its evolutions
$(\boldsymbol{\nu} v)\,(\overline{a}\langle v\rangle \mid \overline{v}\langle t\rangle)$	$(\boldsymbol{\nu} v)\,(\overline{v}\langle t\rangle \mid \overline{a}\langle v\rangle)$	$(\boldsymbol{\nu} v)\overline{a}\langle v\rangle.\overline{v}\langle t\rangle$

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▷ program a particular construct

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- *compositionality* is crucial

  two equivalent systems should be undistinguishable,
  in any context (we are in a concurrent setting)

When are two terms behaviourally equivalent?

- when they "act the same" (?!?) ?

$$\overline{a}\langle v\rangle \mid a(x).\big(\overline{c}\langle x\rangle \mid \overline{d}\langle x\rangle\big) \qquad\qquad (\boldsymbol{\nu} c)\,(\overline{c}\langle c\rangle \mid c(x).\mathbf{0})$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\overline{c}\langle v\rangle \mid \overline{d}\langle v\rangle \qquad\qquad\qquad\qquad \mathbf{0}$$

$$\not\downarrow \qquad\qquad\qquad\qquad \not\downarrow$$

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- does not look very compositional

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let us first concentrate on CCS

Labelled transitions

$$a.P \quad \xrightarrow{a} \quad P$$

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N.B.: a *chemical semantics* for CCS? →rather straightforward

Traces

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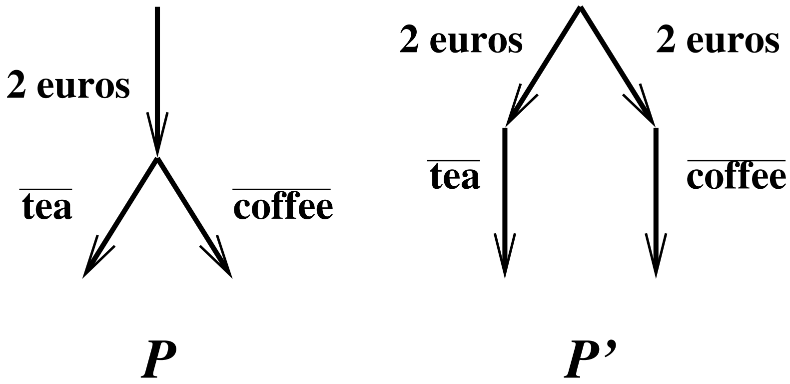
- should we compare traces?

Definition [trace equivalence] P and Q are *trace equivalent* iff they have the same set of traces.

Milner's coffe machines



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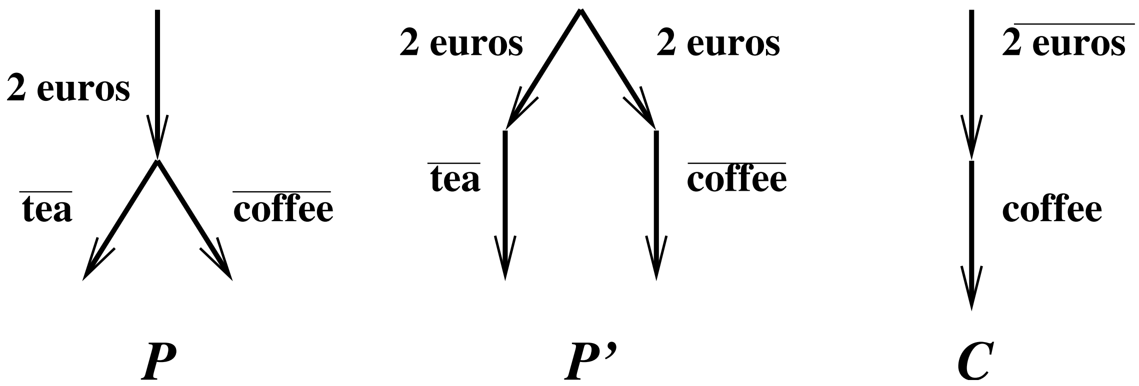

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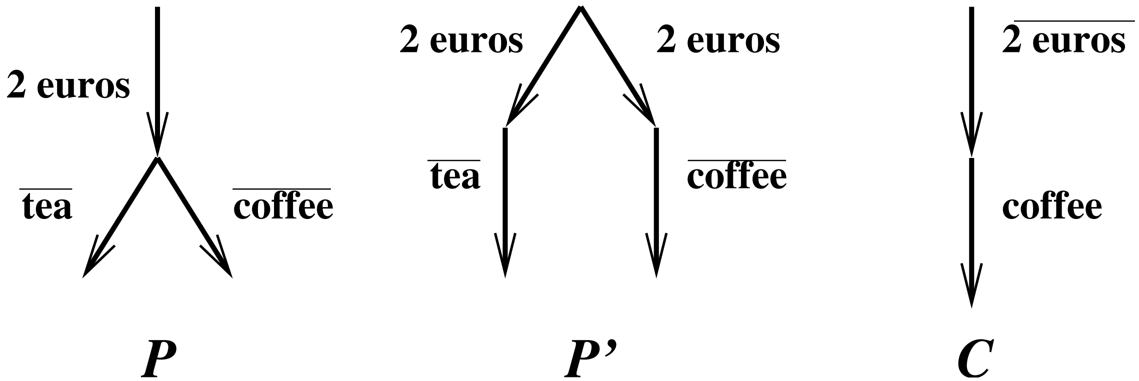

$P' \mid C$ may exhibit a deadlock

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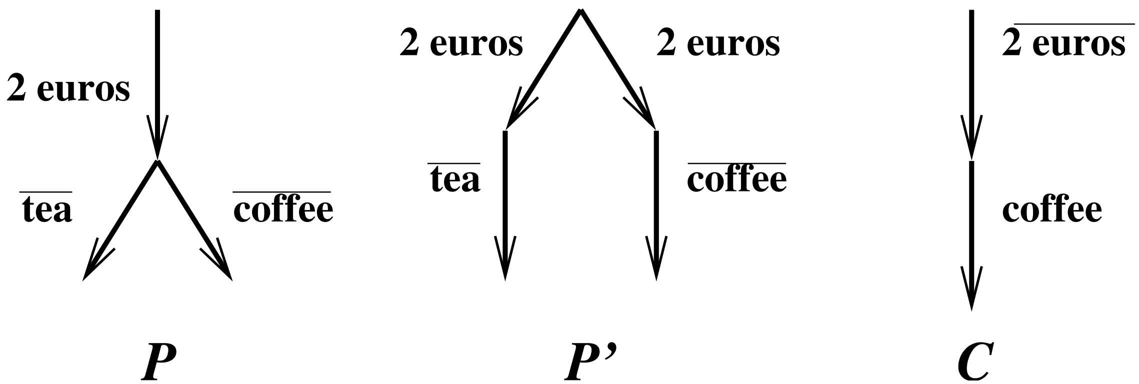

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Trace equivalence is not compositional

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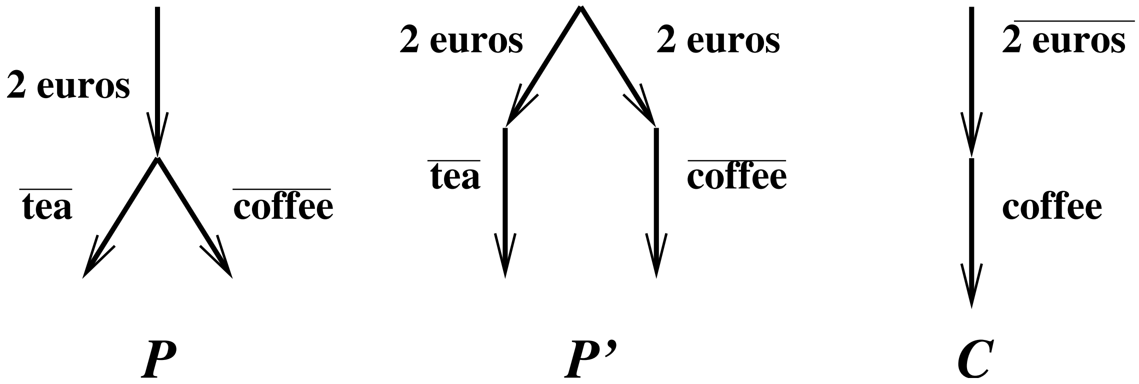

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Trace equivalence is not compositional

one should be less "factual" (see *"Marignan: 1515"*)

(*linear* vs *branching* time)

Towards compositionality

Definition [bisimulation]: A symmetrical relation $\mathcal{R}$ on processes is a *bisimulation* iff, whenever $P \,\mathcal{R}\, Q$, $P \xrightarrow{\mu} P'$ implies that there exists Q s.t. $Q \xrightarrow{\mu} Q'$ and $P' \,\mathcal{R}\, Q'$.

Definition [bisimilarity]: Bisimilarity ($\sim$) is the union of all bisimulations.

Remarks: $\sim$ is an equivalence relation
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Theorem: in CCS, bisimilarity is a congruence.
(and hence it is compositional w.r.t. parallel composition)

Exercise: bisimulation versus two simulations

Definition: Q simulates P if there exists a relation $\mathcal{R}$ s.t. $P\,\mathcal{R}\,Q$ and $P \xrightarrow{\mu} P'$, there exists Q' s.t. $Q \xrightarrow{\mu} Q'$ and $P'\,\mathcal{R}\,Q'$.
$P \leftrightarrows Q$ if Q simulates P and P simulates Q.

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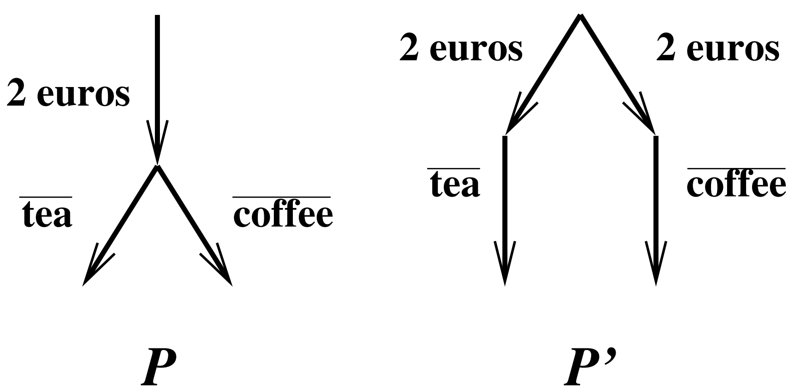

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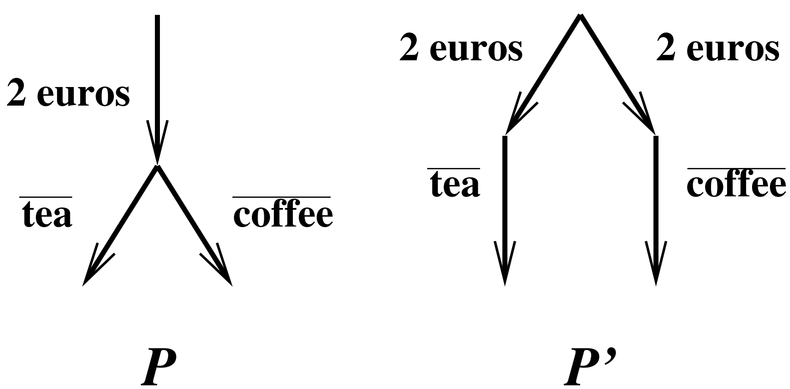

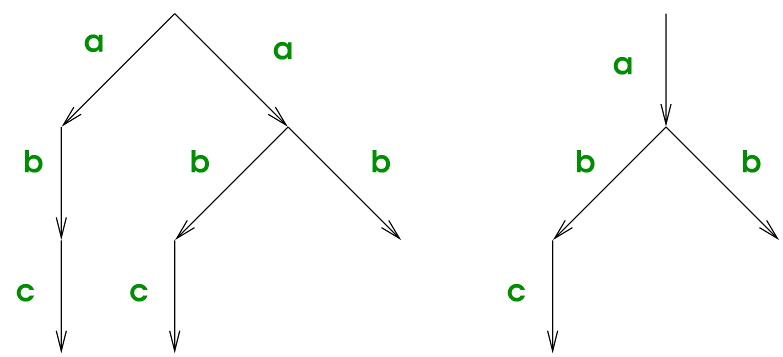

Bisimulation and up-to bisimulation

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- bisimulation:
$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \downarrow \mu \\ P' & \mathcal{R} & Q' \end{array}$$
- a framework for bisimulation proof techniques:

up-to bisimulation

let $\mathcal{F}$ be a function from relations to relations

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \downarrow \mu \\ P' & \mathcal{F}(\mathcal{R}) & Q' \end{array}$$

if $\mathcal{F}$ gives a valid proof technique, then $\mathcal{R} \subseteq \sim$

Exercise – an up-to technique

- $\mathcal{R}$ is a *bisimulation up to bisimilarity* if

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▷ $\mathcal{R}$ itself is not necessarily a bisimulation

▷ useful to "plug" known bisimilarity laws into bisimulation proofs – other such techniques exist

Weak bisimilarity

Bisimulation – weak case

Definition [weak transitions] $\Rightarrow$: refl. trans. closure of $\xrightarrow{\tau}$;

abstract from internal computations

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whenever $P \, \mathcal{R} \, Q$ and $P \xrightarrow{\mu} P'$, there exists Q' s.t. $Q \stackrel{\widehat{\mu}}{\Longrightarrow} Q'$ and $P' \, \mathcal{R} \, Q'$.
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Proposition: $\approx$ is an equivalence relation.

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  consider $a + \tau.(b|c)$ and $a + (b|c)$

$\approx$ and τ

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- ▷ let $A \stackrel{\text{def}}{=} a + \tau.A$, $A \;\approx\; a.\mathbf{0}$

Weak bisimulation up to (weak) bisimilarity

let us try to establish a proof technique similar to (strong) bisimulation up to $\sim$ in the weak case:

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \Downarrow \mu \\ P' & \approx \mathcal{R} \approx & Q' \end{array}$$

and symmetrically, then $\mathcal{R} \subseteq \approx$?

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$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \Downarrow \mu \\ P' & \sim \mathcal{R} \sim & Q' \end{array}$$ is ok, $$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \Downarrow \mu \\ P' & \gtrsim \mathcal{R} \lesssim & Q' \end{array}$$ is also ok

Definition [expansion]: $\mathcal{R}$ is an expansion iff whenever $P \mathcal{R} Q$:

- if $P \xrightarrow{\mu} P'$ there exists Q' s.t. $Q \stackrel{\widehat{\mu}}{\Longrightarrow} Q'$ and $P' \mathcal{R} Q'$
- if $Q \xrightarrow{\mu} Q'$ there exists P' s.t. $P \xrightarrow{\widehat{\mu}} P'$ and $P' \mathcal{R} Q'$.

$\lesssim$ is the greatest expansion, $\gtrsim$ is $\lesssim^{-1}$.

Behavioural equivalences for π

Labelled Transition System for the π-calculus

what are the (labelled) transitions of the following term?

$$(\boldsymbol{\nu} x)(\boldsymbol{\nu} y)\Big(\overline{a}\langle w\rangle.P \mid b(t).Q \mid \overline{y}\langle v\rangle.\mathbf{0} \mid \overline{b}\langle x\rangle.R\Big)$$

what are the possible actions in π?

LTS for the π-calculus

three (+1) kinds of actions: $\begin{cases} P \xrightarrow{a(b)} Q \\ P \xrightarrow{\overline{a}\langle b\rangle} Q, \quad P \xrightarrow{\overline{a}(b)_{\boldsymbol{\nu}}} Q \\ P \xrightarrow{\tau} Q \end{cases}$

names: n(μ)
bound names: bn($\overline{a}(b)$) $= \{b\}$, bn(μ) $= \emptyset$ otherwise

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N.B.: in a polyadic setting, bound outputs are of the form $(\boldsymbol{\nu}\tilde{x})\,\overline{a}\langle\tilde{y}\rangle$, with $\tilde{x} \subseteq \tilde{y}$ and $\tilde{x}$ is a set rather than a tuple

$\rightarrow$ *a precise, rigorous definition is really tedious*

Labelled transitions for the π-calculus, the rules

$$\text{Inp} \quad a(m).P \xrightarrow{a(n)} P_{\{m \leftarrow n\}} \qquad \text{Out} \quad \overline{a}\langle n \rangle.P \xrightarrow{\overline{a}\langle n \rangle} P$$

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symmetrical versions of rules Comm$_\text{l}$, Par$_\text{l}$ *and* Close$_\text{l}$ *have been omitted*

Labelled transitions: a derivation

$$\frac{}{(\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q) \mid R \mid a(x).S \;\longrightarrow}$$

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$$\text{Par}\ \dfrac{}{}$$

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$$
\text{Par}\ \dfrac{\text{Open}\ \dfrac{\text{Par}\ \dfrac{\text{Out}\quad \overline{a}\langle c\rangle.Q \xrightarrow{\overline{a}\langle c\rangle} Q}{P \mid \overline{a}\langle c\rangle.Q \xrightarrow{\overline{a}\langle c\rangle} P \mid Q}}{(\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q) \xrightarrow{\overline{a}(c)} P \mid Q}}{}
$$

$$
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$$

$$
\text{Close}\ \dfrac{\triangle \qquad\qquad \text{Inp}\quad a(x).S \xrightarrow{a(c)} S_{\{x\leftarrow c\}}}{(\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q) \mid R \mid a(x).S \xrightarrow{\tau} (\boldsymbol{\nu} c)\big(P \mid Q \mid R \mid S_{\{x\leftarrow c\}}\big)}
$$

Same computation in chemical version

$$\begin{aligned}
(\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q) \mid R \mid a(x).S &\equiv (\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q \mid R) \mid a(x).S \\
&\equiv (\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q \mid R \mid a(x).S) \\
&\equiv (\boldsymbol{\nu} c)\,(a(x).S \mid \overline{a}\langle c\rangle.Q \mid P \mid R) \qquad \triangle_1
\end{aligned}$$

$$(\boldsymbol{\nu} c)\,(S_{\{x\leftarrow c\}} \mid Q \mid P \mid R) \equiv (\boldsymbol{\nu} c)\,(P \mid Q \mid R \mid S_{\{x\leftarrow c\}}) \qquad \triangle_2$$

$$\triangle_1\ \triangle_2\quad \dfrac{\dfrac{\dfrac{a(x).S \mid \overline{a}\langle c\rangle.Q \longrightarrow S_{\{x\leftarrow c\}} \mid Q}{a(x).S \mid \overline{a}\langle c\rangle.Q \mid P \mid R \longrightarrow S_{\{x\leftarrow c\}} \mid Q \mid P \mid R}}{(\boldsymbol{\nu} c)\,(a(x).S \mid \overline{a}\langle c\rangle.Q \mid P \mid R) \longrightarrow (\boldsymbol{\nu} c)\,(S_{\{x\leftarrow c\}} \mid Q \mid P \mid R)}}{(\boldsymbol{\nu} c)\,(P \mid \overline{a}\langle c\rangle.Q) \mid R \mid a(x).S \longrightarrow (\boldsymbol{\nu} c)\,(P \mid Q \mid R \mid S_{\{x\leftarrow c\}})}$$

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$\longrightarrow$ *:* is easier to read;
work modulo α-conversion, AC properties of $|, +$
and moving $\boldsymbol{\nu}$ around

$\xrightarrow{\mu}$: manipulate *syntax trees*; the "redex" is read *"on the term"*

progressively construct the interaction between a term
and its context

Porting the definition of bisimilarity in the π-calculus

- same thing as before:

bisimulation: $$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \downarrow \mu \\ P' & \mathcal{R} & Q' \end{array}$$ $\sim$ is the greatest bisimulation

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▷ why this does not happen in CCS?

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▷ why this does not happen in CCS?

▷ we have though: $(\boldsymbol{\nu} b)\,(\overline{a} \mid b) \;\sim^c\; (\boldsymbol{\nu} b)\,(\overline{a}.b + b.\overline{a})$,
$\sim^c$ being the greatest congruence included in $\sim$

Bisimliarity – some example laws

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Expansion lemma

Lemma [expansion]: if $M = \alpha_1.P_1 + \cdots + \alpha_n.P_n$ and $N = \beta_1.Q_1 + \cdots + \beta_m.Q_m$ then

$$M \mid N \sim \Sigma_i \alpha_i.(P_i|N) + \Sigma_j \beta_j.(M|Q_j) + \Sigma_{\langle \alpha_i \text{ comp } \beta_j \rangle} \tau.R_{ij}$$

with α_i comp β_j (α_i is the "dual" of β_j):

$\alpha_i = \overline{x}\langle y \rangle$ and $\beta_j = x(z)$, in which case $R_{ij} = P_i|Q_{j\{y \leftarrow z\}}$, or symmetrically.

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let us prove that in the π-calculus, $T \,|\, U \;\sim\; U \,|\, T$ for all T, U

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A bisimulation proof

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... no, with $\{\big((\boldsymbol{\nu} \tilde{x})\,(P|Q),\, (\boldsymbol{\nu} \tilde{x})\,(Q|P)\big)\}$ ($\tilde{x}$: set of names)

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  useful with replication

An important law about replications

processes of the form $!a(x).P$ may be seen as *resources*

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$\mathcal{R}$ *is a bisimulation up to bisimilarity, up to restriction and up to parallel composition*

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- one should observe *possibilities of interaction*: barbs

- $P \downarrow_a$ (resp. $P \downarrow_{\overline{a}}$): P may receive (resp. emit) on a

"I can offer coffee or tea"

Remark: $P \downarrow_a \quad \Leftrightarrow \quad P \equiv (\boldsymbol{\nu}\tilde{v})\,(a(x).R \;\mid\; T)$, $a \notin \tilde{v}$

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BUT **Theorem:** $\dot{\sim}^c$, the greatest congruence included in $\dot{\sim}$, coincides with $\sim^c$

$\sim$ is "$\forall R.\, P \,|\, R \;\dot{\sim}\; Q \,|\, R$"

When labelled transitions coincide with barbs

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- ▷ $\mathcal{C}, \mathcal{C}'s$ are chosen so that this entails that $Q \xrightarrow{\mu} Q'$

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. . . and other notions of bisimilarity, e.g. in *asynchronous* π

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Theorem: $\sim_{\text{late}} \subsetneq \sim$. Proof: $P \stackrel{\text{def}}{=} x(z) + x(z).\overline{z}$

$Q \stackrel{\text{def}}{=} x(z) + x(z).\overline{z} + x(z).[z = y]\overline{z}$