CIS 301: Logical Foundations of Programming, Exam 1

October 16, 2002

General Notes

- Open textbook (Huth & Ryan), open class notes, open solutions of homework assignments.
- Please write your name on this page.

Good Luck!

MY NAME:

1. *7 points.* Prove the following sequent

$$\neg(p \wedge q) \wedge (r \vee p) \vdash \neg r \rightarrow \neg q$$

using only the rules presented in Huth & Ryan's Fig. 1.2, p. 31.
(In particular, you cannot use the "proved equivalences" given on p. 34.)

2. *6 points.* Consider the sequent you have just proven in Question 1. By soundness (Theorem 1.34), it must be that we indeed have

$$\neg(p \wedge q) \wedge (r \vee p) \models \neg r \rightarrow \neg q$$

Please complete the proof skeleton given below.

(Remember that $[\![\phi]\!]$ denotes the truth value of ϕ under a given assignment of truth values to atoms.)

Proof: Let $\phi_1 = \neg(p \wedge q) \wedge (r \vee p)$ and let $\phi_2 = \neg r \rightarrow \neg q$. We must show (Definition 1.33) that if $[\![\phi_1]\!] = T$ then $[\![\phi_2]\!] = T$. Towards a contradiction, assume that $[\![\phi_1]\!] = T$ but $[\![\phi_2]\!] = F$. That is,

$$[\![\neg(p \wedge q) \wedge (r \vee p)]\!] = T \tag{1}$$
$$[\![\neg r \rightarrow \neg q]\!] = F \tag{2}$$

By truth tables, (1) is only possible if

$$[\![\neg(p \wedge q)]\!] = T \tag{3}$$
$$[\![r \vee p]\!] = T \tag{4}$$

By truth tables, (2) is only possible if

$$\begin{aligned} [\![\neg r]\!] &= T \\ [\![\neg q]\!] &= F \end{aligned}$$

YOUR TURN...

3. *5 points.* We recursively define the size of a formula ϕ, denoted $\mathrm{S}(\phi)$, as follows:

$$
\begin{aligned}
\mathrm{S}(p) &= 1 \\
\mathrm{S}(\neg\phi) &= \mathrm{S}(\phi) + 1 \\
\mathrm{S}(\phi_1 \wedge \phi_2) &= \mathrm{S}(\phi_1) + \mathrm{S}(\phi_2) + 1 \\
\mathrm{S}(\phi_1 \vee \phi_2) &= \mathrm{S}(\phi_1) + \mathrm{S}(\phi_2) + 1 \\
\mathrm{S}(\phi_1 \rightarrow \phi_2) &= \mathrm{S}(\phi_1) + \mathrm{S}(\phi_2) + 1
\end{aligned}
$$

Recall the definition of the function NNF on page 80 in Huth & Ryan. Even though the use of NNF often decreases the size of a formula, it may happen that the use of NNF actually *increases* the size of a formula. That is, there exists a formula ϕ such that

$$\mathrm{S}(\mathrm{NNF}(\phi)) > \mathrm{S}(\phi)$$

Your task is to write down such a ϕ.

(You must also indicate the values of $\mathrm{NNF}(\phi)$, $\mathrm{S}(\mathrm{NNF}(\phi))$, and $\mathrm{S}(\phi)$.)

4. *7 points.* In Question 3, we saw that NNF may increase the size of a formula. But NNF never doubles the size of a formula:

Theorem: For all ϕ, $\mathrm{S}(\mathrm{NNF}(\phi)) < 2\,\mathrm{S}(\phi)$.

Please complete the following proof skeleton.

Proof: With

$$\begin{aligned}\mathrm{LHS}_\phi &= \mathrm{S}(\mathrm{NNF}(\phi))\\ \mathrm{RHS}_\phi &= 2\,\mathrm{S}(\phi)\end{aligned}$$

our task is to prove that for all ϕ

$$\mathrm{LHS}_\phi < \mathrm{RHS}_\phi.$$

The proof is by induction in the size of ϕ, and it is enough to consider the cases below (as the other cases are symmetric).

- ϕ is a literal: Since $\mathrm{LHS}_\phi > 0$ we infer the desired
$$\mathrm{LHS}_\phi < 2\,\mathrm{LHS}_\phi = 2\,\mathrm{S}(\mathrm{NNF}(\phi)) = 2\,\mathrm{S}(\phi) = \mathrm{RHS}_\phi.$$

- ϕ takes the form $\neg\neg\phi_1$: then
$$\begin{aligned}\mathrm{LHS}_\phi &= \mathrm{S}(\mathrm{NNF}(\neg\neg\phi_1)) = \mathrm{S}(\mathrm{NNF}(\phi_1)) = \mathrm{LHS}_{\phi_1}\\ \mathrm{RHS}_\phi &= 2\,\mathrm{S}(\neg\neg\phi_1) = 2\,(\mathrm{S}(\phi_1) + 2) = \mathrm{RHS}_{\phi_1} + 4\end{aligned}$$
Inductively, we can assume that $\mathrm{LHS}_{\phi_1} < \mathrm{RHS}_{\phi_1}$. Therefore,
$$\mathrm{LHS}_\phi = \mathrm{LHS}_{\phi_1} < \mathrm{RHS}_{\phi_1} + 4 = \mathrm{RHS}_\phi.$$

YOUR TURN...

- ϕ takes the form $\phi_1 \land \phi_2$:

- ϕ takes the form ________________:

Errata. In the original exam, there was an unfortunate typo in Question 4, where line 15 instead of

$$2\,(\mathrm{S}(\phi_1) + 2)$$

had the ill-formed

$$2\,(\mathrm{S}(\phi_1 + 2))$$