CIS 301: Logical Foundations of Programming, Exam 1

March 8, 2002

General Notes

- Open textbook (Huth and Ryan), open class notes, open solutions of homework assignments.
- Please write your name on this page.

Good Luck!

1. *15 points*. Prove the following sequent using the proof rules studied in class:

$$(p \to q) \to q \vdash p \vee q$$

2. *10 points*. Prove that

$$p \vee q \models (p \to q) \to q$$

3. *15 points*. Consider the following BNF of *terms*:

$$t \quad ::= \quad \mathsf{true} \mid \mathsf{false} \mid \mathsf{0} \mid \mathsf{succ}\ t \mid \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$$

As we have seen in class, "succ" stands for the successor function. Define the set of constants appearing in a term t, written $\mathit{Consts}(t)$ as follows:

$$\begin{array}{lcl}
\mathit{Consts}(\mathsf{true}) & = & \{\mathsf{true}\} \\
\mathit{Consts}(\mathsf{false}) & = & \{\mathsf{false}\} \\
\mathit{Consts}(0) & = & \{0\} \\
\mathit{Consts}(\mathsf{succ}\ t_1) & = & \mathit{Consts}(t_1) \\
\mathit{Consts}(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) & = & \mathit{Consts}(t_1) \cup \mathit{Consts}(t_2) \cup \mathit{Consts}(t_3)
\end{array}$$

Define the size of a term t, written $\mathit{size}(t)$ as follows:

$$\begin{array}{lcl}
\mathit{size}(\mathsf{true}) & = & 1 \\
\mathit{size}(\mathsf{false}) & = & 1 \\
\mathit{size}(0) & = & 1 \\
\mathit{size}(\mathsf{succ}\ t_1) & = & \mathit{size}(t_1) + 1 \\
\mathit{size}(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) & = & \mathit{size}(t_1) + \mathit{size}(t_2) + \mathit{size}(t_3) + 1
\end{array}$$

Let $|\mathit{Consts}(t)|$ be the number of elements in the set $\mathit{Consts}(t)$.

Prove that for all terms t, $|\mathit{Consts}(t)| \leq \mathit{size}(t)$.

4. *10 points*. We would like to prove the following (incorrect) claim: For any n-element set, all elements of the set are identical. (An n-element set is a set that contains n elements).

   Let $P(n)$ be defined as "all elements in an n-element set, S, are identical". Your job in this problem is to find the bug in the following inductive "proof".
   Proof: $P(1)$ holds because all elements in a 1-element set are identical. There is only 1 element in the set and clearly the element is identical to itself.
   Induction hypothesis: Assume $P(n)$.
   Inductive step: Show $P(n+1)$, *i.e.*, show that if S contains $n+1$ elements, then all elements in S are identical.
   Let us write S as $S_1 \cup S_2$ where S_1 and S_2 each have n elements and each contains a. By the induction hypothesis on n, all elements in S_1 are identical to a and all elements in S_2 are identical to a. Hence all elements in S are identical to a. END OF PROOF.

   N.B.: It is *not* sufficient to exhibit a set, *e.g.*, $\{1, 2, 3\}$ and to note that all its elements are different. You must find the bug in the argument in the above "proof" itself.