CIS 301: Logical Foundations of Programming, Exam 2

April 12, 2002

General Notes

- Open textbook (Huth and Ryan), open class notes, open solutions of homework assignments.
- Please write your name on this page.
- Use both sides of a page if necessary.
- *Please read all the questions first.*

Good Luck!

1. *15 points*. Prove the following sequent using the proof rules studied in class:

$$\neg\exists x(P(x) \wedge Q(x)), \forall x(R(x) \rightarrow P(x)) \vdash \neg\exists x(R(x) \wedge Q(x))$$

2. *20 points*. As we have seen in class, a list of integers can be defined using the following BNF:

$$\ell \ ::= \ [\,] \mid n \ :: \ \ell$$

That is, a list of integers is either an empty list or a nonempty list that can be constructed by inserting an integer n to the front of a list of integers ℓ; this is what is represented by the notation $n \ :: \ \ell$.

Consider the function sumlist that returns the sum of all the elements in a list of integers. Consider also the function double that doubles every element in a list of integers. Thus $\mathsf{sumlist}([1;2;5]) = 8$, and $\mathsf{double}([1;2;5]) = [2;4;10]$. Here are the definitions of the two functions:

$$\begin{aligned} \mathsf{sumlist}([\,]) &= 0 \\ \mathsf{sumlist}(n :: \ell) &= n + \mathsf{sumlist}(\ell) \end{aligned}$$

$$\begin{aligned} \mathsf{double}([\,]) &= [\,] \\ \mathsf{double}(n :: \ell) &= (2 * n) :: \mathsf{double}(\ell) \end{aligned}$$

Prove that for any list ℓ, $\mathsf{sumlist}(\mathsf{double}(\ell)) = 2 * \mathsf{sumlist}(\ell)$. Remember to write down what you are doing induction on as well as the induction hypothesis.

3. *15 points*. Say that a formula ϕ is in *disjunctive normal form* (DNF) if

$$\phi = \psi_1 \vee \psi_2 \vee \ldots \vee \psi_n$$

where, for $i = 1, \ldots, n$, each ψ_i is a conjunction of literals: $\psi_i = L_{i_1} \wedge \ldots \wedge L_{i_k}$.

Given any implication-free, negation normal form formula ϕ as input, the function $\mathsf{dnf}(\phi)$ that converts ϕ into DNF, is defined as follows:

$$\begin{array}{rcl}
\mathsf{dnf}(\phi) & = & \phi \text{ if } \phi \text{ is a literal} \\
\mathsf{dnf}(\phi_1 \vee \phi_2) & = & \mathsf{dnf}(\phi_1) \vee \mathsf{dnf}(\phi_2) \\
\mathsf{dnf}(\phi_1 \wedge \phi_2) & = & \mathsf{DISTR}(\mathsf{dnf}(\phi_1), \mathsf{dnf}(\phi_2))
\end{array}$$

$$\begin{array}{rcl}
\mathsf{DISTR}(\eta_1, \eta_2) & = & \text{// } \eta_1, \eta_2 \text{ are in DNF} \\
\textbf{case } \eta_1 = \eta_{11} \vee \eta_{12} : & \textbf{return} & \mathsf{DISTR}(\eta_{11}, \eta_2) \vee \mathsf{DISTR}(\eta_{12}, \eta_2) \\
\textbf{case } \eta_2 = \eta_{21} \vee \eta_{22} : & \textbf{return} & \mathsf{DISTR}(\eta_1, \eta_{21}) \vee \mathsf{DISTR}(\eta_1, \eta_{22}) \\
\text{otherwise, no disjunctions:} & \textbf{return} & \eta_1 \wedge \eta_2
\end{array}$$

Our goal in this question is to prove that for any implication-free, negation normal form formula ϕ, $[\![\mathsf{dnf}(\phi)]\!] = [\![\phi]\!]$.

(a) *7 points*. What property must $[\![\mathsf{DISTR}(\eta_1, \eta_2)]\!]$ satisfy?

(b) *8 points*. Assume the property in part (a), without proof. Now prove that $[\![\mathsf{dnf}(\phi)]\!] = [\![\phi]\!]$.

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