Scott-Domains, Generalized Ultrametric Spaces and Generalized Acyclic Logic Programs

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Abstract

Every Scott-domain can be viewed as a generalized ultrametric space with properties which allow to apply a generalization of the Banach contraction mapping theorem. We will give this construction in detail and apply it to a class of programs which is stricly larger than the class of all acyclic programs. The paper is taken from [Hit98, Chapter 4].

1 Domains as Generalized Ultrametric Spaces

We first introduce Scott-domains and generalized ultrametric spaces. The following is taken from [SLG94].

1.1 Definition (Scott-Ershov-domain) A partially ordered set A is called *consistent* if it has an upper bound, and is called *directed* if every finite subset of A has an upper bound in A. A partial ordered set $(D, \leq)$ is called a *complete partial order* (*cpo*) if

(1) there exists $\bot \in D$ such that for all $a \in D$ we have $\bot \leq a$ ($\bot$ is called the *bottom element* of D) and

(2) if $A \subseteq D$ is a directed set, then $\sup A$ exists in D.

An element c of a cpo is called *compact* (or *finite*) if, for every directed set $A \subseteq D$ with $c \leq \sup A$, there is some $a \in A$ with $c \leq a$. We denote the set of all compact elements of D by D_c.
A cpo D is called a (*Scott-Ershov-*) *domain* if

(1) for every $a \in D$ the set $\mathrm{approx}(a) := \{c \in D_c \mid c \leq a\}$ is directed, $a = \sup \mathrm{approx}(a)$ (D is *algebraic*) and

(2) every consistent set in D has a supremum (D is *consistently complete*).

Intuitively, $x \leq y$ in a domain can be interpreted as "x approximates y". Compact elements can be considered as practically implementable objects in a computer system, so that every object of interest can be arbitrarily closely approximated by those.
The following is taken from [PR97].

1.2 Definition (generalized ultrametric space) Let X be a set and let Γ be a partial order with least element 0. We call (X, d) a *generalized ultrametric space* if $d : X \times X \to \Gamma$ is a function such that for all $x, y, z \in X$

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$, and

(3) if $d(x, y), d(y, z) \leq \gamma$ then $d(x, z) \leq \gamma$.

For $0 \neq \gamma \in \Gamma$ and $x \in X$, the set $B_\gamma(x) := \{y \in X \mid d(x, y) \leq \gamma\}$ is called a (γ-)ball in X. A generalized ultrametric space is called *spherically complete* if for any chain $(\mathcal{C}, \subseteq)$ of balls in X, $\bigcap \mathcal{C} \neq \emptyset$.
A function $f : X \to X$ is called

(1) *contractive* if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$,

(2) *strictly contracting on orbits* if $d(f^2(x), f(x)) < d(f(x), x)$ for every $x \in X$ with $x \neq f(x)$, and

(3) *strictly contracting* if $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

We will need the following observations, which are well-known for ultrametric spaces.

1.3 Lemma Let (X, d, Γ) be a generalized ultrametric space. For $\alpha, \beta \in \Gamma$ and $x, y \in X$ the following statements hold.

(1) If $\alpha \leq \beta$ and $B_\alpha(x) \cap B_\beta(y) \neq \emptyset$, then $B_\alpha(x) \subseteq B_\beta(y)$.

(2) If $B_\alpha(x) \cap B_\alpha(y) \neq \emptyset$, then $B_\alpha(x) = B_\alpha(y)$.

(3) $B_{d(x,y)}(x) = B_{d(x,y)}(y)$.

Proof: Let $a \in B_\alpha(x)$ and $b \in B_\alpha(x) \cap B_\beta(y)$. Then $d(a, x) \leq \alpha$ and $d(b, x) \leq \alpha$, hence $d(a, b) \leq \alpha \leq \beta$. Since $d(b, y) \leq \beta$, we have $d(a, y) \leq \beta$, hence $a \in B_\beta(y)$, which proves the first statement. The second follows by symmetry and the third by replacing α by $d(x, y)$. ■

The following theorem was given in [PR97] in a more general form.

1.4 Theorem (Priess-Crampe and Ribenboim) Let (X, d) be a spherically complete generalized ultrametric space and let $f : X \to X$ be contractive and strictly contracting on orbits. Then f has a fixed point. Moreover, if f is strictly contracting on X, then f has a unique fixed point.

Proof: Assume that f has no fixed point. Then $d(x, f(x)) \neq 0$ for all $x \in X$. We define the set $\mathcal{B} := \{B_{d(x,f(x))}(x) \mid x \in X\}$. Now let $\mathcal{C}$ be a maximal chain in $\mathcal{B}$. Since X is spherically complete, there exists $z \in \bigcap \mathcal{C}$. We show, that $B_{d(z,f(z))} \subseteq \bigcap \mathcal{C}$. Let $B_{d(x,f(x))}(x) \in \mathcal{C}$. Since $z \in B_{d(x,f(x))}(x)$, we get $d(z, x) \leq d(x, f(x))$ and $d(z, f(x)) \leq d(x, f(x))$. By non-expansiveness of f, we get $d(f(z), f(x)) \leq d(z, x) \leq d(x, f(x))$. It follows, that $d(z, f(z)) \leq d(x, f(x))$ and therefore $B_{d(z,f(z))}(z) \subseteq B_{d(x,f(x))}(x)$ by Lemma 1.3. Since x was chosen arbitrarily, $B_{d(z,f(z))}(z) \subseteq \bigcap \mathcal{C}$.
Now since f is strictly contracting on orbits, $d(f(z), f^2(z)) < d(z, f(z))$, and therefore $z \notin B_{d(f(z),f^2(z))}(f(z)) \subset B_{d(z,f(z))}(f(z))$. By Lemma 1.3, this is equivalent to $B_{d(f(z),f^2(z))}(f(z)) \subset B_{d(z,f(z))}(z)$, which is a contradiction to the maximality of $\mathcal{C}$. So f has a fixed point.
Now let f be strictly contracting on X and assume that x, y are two distinct fixed points of f. Then we get $d(x, y) = d(f(x), f(y)) < d(x, y)$ which is impossible. So the fixed point of f is unique in this case. ■

Note that the above given proof is not constructive, so it does not indicate a means by which one can actually find a fixed point.
In order to apply this result, we show first how every domain can be viewed as a spherically complete generalized ultrametric space. For some countable ordinal γ, let Γ_γ be the set $\{2^{-\alpha} \mid \alpha < \gamma\}$ of symbols $2^{-\alpha}$ with ordering $2^{-\alpha} < 2^{-\beta}$ if and only if $\beta < \alpha$.

1.5 Definition (see [SH97]) Let D be a domain and $r : D_c \to \gamma$ a function, called a *rank function*, and denote $2^{-\gamma}$ by 0. Define $d_r : D \times D \to \Gamma_{\gamma+1}$ by

$$d_r(x, y) := \inf\{2^{-\alpha} \mid c \leq x \text{ if and only if } c \leq y \text{ for every } c \in D_c \text{ with } r(c) < \alpha\}.$$

Then (D, d_r) is called the generalized ultrametric space *induced by* r.

It is straightforward to see, that (D, d_r) is indeed a generalized ultrametric space.
We proceed to show, that (D, d_r) is spherically complete. For every generalized ultrametric which is induced by some rank function, we will denote the ball $B_{2^{-\alpha}}(x)$ in the following by $B_\alpha(x)$.

1.6 Lemma (see [SH97]) Let $B_\alpha(x) \subseteq B_\beta(y)$ (so $\beta \leq \alpha$). Then the following statements hold.

(1) $\{c \in \text{approx}(x) \mid r(c) \leq \beta\} = \{c \in \text{approx}(y) \mid r(c) \leq \beta\}$.

(2) $B_\alpha := \sup\{c \in \text{approx}(x) \mid r(c) \leq \alpha\}$ and $B_\beta := \sup\{c \in \text{approx}(y) \mid r(c) \leq \beta\}$ both exist.

(3) $B_\beta \leq B_\alpha$.

Proof: Since $d_r(x, y) \leq 2^{-\beta}$, the first statement follows immediately from the definition of d_r. The second statement follows from the fact that every domain is consistently complete. The third statement follows from the observation that $B_\beta = \sup\{c \in \text{approx}(y) \mid r(c) \leq \beta\} = \sup\{c \in \text{approx}(x) \mid r(c) \leq \beta\} \leq \sup\{c \in \text{approx}(x) \mid r(c) \leq \alpha\} = B_\alpha$. ■

1.7 Theorem (see [SH97]) (D, d_r) is spherically complete.

Proof: By the previous lemma, every chain $(B_\alpha(x_\alpha))$ of balls in D gives rise to a chain (B_α) in D in reverse order. Let $B := \sup B_\alpha$. Now let $B_\alpha(x)$ be an arbitrary ball in the chain. It suffices to show that $B \in B_\alpha(x)$. Since $B_\alpha \in B_\alpha(x)$, we have $d_r(B_\alpha, x) \leq 2^{-\alpha}$, and since d_r is an ultrametric, it remains to show that $d_r(B, B_\alpha) \leq 2^{-\alpha}$. For every $c \leq B_\alpha$, we have $c \leq B$ by construction of B. Now let $c \leq B$ with $c \in D_c$ and $r(c) < \alpha$. We have to show that $c \leq B_\alpha$. Since D is a domain, hence an algebraic cpo, there exists B_β in the chain with $c \leq B_\beta$. Now suppose $B_\beta \geq B_\alpha$ (otherwise $c \leq B_\alpha$ immediately). Then by the above lemma and the fact that the collection $(B_\alpha(x_\alpha))$ is a chain, we have $B_\beta(x_\beta) \subseteq B_\alpha(x_\alpha)$ and therefore $c \in \{c \in \text{approx}(x_\beta) \mid r(c) \leq \alpha\} = \{c \in \text{approx}(x_\alpha) \mid r(c) \leq \alpha\}$. Since B_α is the supremum of the right-hand set, $c \leq B_\alpha$. ∎

It should be noted that we needed both algebraicity and consistent completeness of domains to prove the previous theorem.

2 Application to Generalized Acyclic Logic Programs

We apply this result to logic programming. We next introduce level mappings on I_P, which will be used for defining rank functions. For the following, we denote the set of all finite subsets of I_P, which is the set of all compact elements in I_P, by I_c.

2.1 Definition (level mapping) Let P be a normal logic program and let $\gamma \in \Omega$. A mapping $l : B_P \to \gamma$ is called a *level mapping*. We call l an ω-level mapping if $\gamma = \omega$. We set $\mathcal{L}_\alpha := \{A \in B_P \mid l(A) < \alpha\}$ for $\alpha \leq \gamma$ and $\mathcal{L}_0 = \emptyset$.
We define the rank function *induced by* the level mapping l by $r(I) := \max\{l(A) \mid A \in I\}$ for every $I \in I_c$. A generalized ultrametric obtained by such a rank function will further be denoted by d_l.

The following proposition makes calculation of distances easier.

2.2 Proposition Let P be a normal logic program, let l be a level mapping for P and let $I, J \in I_P$. Then $d_l(I, J) = \inf\{2^{-\alpha} \mid I \cap \mathcal{L}_\alpha = J \cap \mathcal{L}_\alpha\}$.

Proof: Immediate by the observation that for every $I \in I_P$, $I = \sup\{\{A\} \mid A \in I\}$. ∎

The results obtained so far will be applied to the semantics of a class of programs which is introduced next.

2.3 Definition (see [SH97]) Let P be a normal logic program. We call P *generalized acyclic* if there exists a level mapping l such that for every clause $H \leftarrow B_1, \ldots, B_{n_1}, \neg C_1, \ldots, \neg C_{n_2}$ in ground(P) $l(B_i) < l(H)$ and $l(C_j) < l(H)$ hold for every $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$.

Acyclic programs, i.e. programs with the property given in the previous definition, where the level mapping is an ω-level mapping, were studied in [AP93] in the context of termination problems. If we weaken the "$<$"-condition in Definition 2.3 to "$\leq$" for positive body literals, we obtain *locally stratified programs* as introduced in [Prz88]. It was shown in [Fit94] that acyclic programs have a unique supported model. We will see that this in fact carries over to generalized acyclic programs. Note that locally stratified programs in general do have more than one supported model since every definite program is locally stratified.

2.4 Theorem (see [SH97]) Let P be a generalized acyclic program with respect to a level mapping l. Then T_P is strictly contracting on (I_P, d_l).

Proof: Let $I_1, I_2 \in I_P$ with $d(I_1, I_2) = 2^{-\alpha}$.
(1) Let $\alpha = 0$, so I_1 and I_2 differ on some element of B_P with level 0. Let $A \in T_P(I_1)$ with $l(A) = 0$. Since P is generalized acyclic, A must be the head of a clause in ground(P) and so $A \in T_P(I_2)$. By the same argument, if $A \in T_P(I_2)$ with $l(A) = 0$, then $A \in T_P(I_1)$. So $T_P(I_1) \cap \mathcal{L}_1 = T_P(I_2) \cap \mathcal{L}_1$, and it follows that

$$d(T_P(I_1), T_P(I_2)) \leq 2^{-1} < 2^{-0} = d(I_1, I_2)$$

as required.
(2) Let $\alpha > 0$, so I_1 and I_2 differ on some element of B_P with level α but agree on all ground atoms of lower level. Let $A \in T_P(I_1)$ with $l(A) \leq \alpha$. Then there is a clause $A \leftarrow A_1, \dots, A_{k_1}, \neg B_1, \dots, \neg B_{l_1} \in \text{ground}(P)$, where $k_1, l_1 \geq 0$, with $A_k \in I_1$ and $B_l \notin I_1$ for all $k = 1, \dots, k_1$, $l = 1, \dots, l_1$. Since P is generalized acyclic and $I_1 \cap \mathcal{L}_\alpha = I_2 \cap \mathcal{L}_\alpha$, it follows that $A_k \in I_2$ and $B_l \notin I_2$ for $k = 1, \dots, k_1$, $l = 1, \dots, l_1$. Therefore, $A \in T_P(I_2)$. By the same argument, if $A \in T_P(I_2)$ with $l(A) = 0$, then $A \in T_P(I_1)$. So $T_P(I_1) \cap \mathcal{L}_{\alpha+1} = T_P(I_2) \cap \mathcal{L}_{\alpha+1}$, and it follows that

$$d(T_P(I_1), T_P(I_2)) \leq 2^{-(\alpha+1)} < 2^{-\alpha} = d(I_1, I_2)$$

as required. ■

2.5 Theorem (see [SH97]) Let P be a generalized acyclic logic program. Then T_P has a unique fixed point and hence P has a unique supported model.

Proof: Immediate by Theorem 1.4 and the previous theorems. ■

2.6 Program Consider the following program P:

$$\begin{aligned} q(0) &\leftarrow \neg p(X), \neg p(s(X)) \\ p(0) &\leftarrow \\ p(s(X)) &\leftarrow \neg p(X) \end{aligned}$$

Define $l : B_P \to \omega + 1$ by $l(p(s^n(0))) = n$ and $l(q(s^n(0))) = \omega$ as a level mapping. By Theorem 2.5, P has a unique supported model which is the set $\{p(s^{2n}(0)) \mid n \in \mathbb{N}\}$.

2.7 Program Let P be the following program:

$$\begin{aligned} p(0,0) &\leftarrow \\ p(s(Y),0) &\leftarrow \neg p(Y,X), \neg p(Y,s(X)) \\ p(Y,s(X)) &\leftarrow \neg p(Y,X) \end{aligned}$$

Define a level mapping on B_P by $l(p(s^k(0), s^j(0))) = \omega k + j$. Then P is strictly level-decreasing and hence has a unique supported model which turns out to be $\{p(0, s^{2n}(0)) \mid n \in \mathbb{N}\} \cup \{p(s^{n+1}(0), s^{2k+1}(0)) \mid k, n \in \mathbb{N}\}$.

Note that Theorem 1.4 only yields the existence of a unique model for generalized acyclic programs. Its proof does not provide a method for actually finding it. In [SH97], such a method is given and is expanded on in [Hit97]. Again in [Hit97], computational adequacy of generalized acyclic programs is studied.

3 Summary

We have seen that the theorem of Priess-Crampe and Ribenboim can be applied to logic programming semantics. In fact, it was possible to show that generalized acyclic programs have a unique supported model. Therefore, all the standard approaches to logic programming semantics coincide for these programs. Furthermore, we have seen that there is a relationship between domains and generalized ultrametric spaces comparable with the relationship between domains and quasi-metrics as studied in [Smy91].

4 Problems

Problem 1 To what extent can the construction of generalized ultrametric spaces out of domains, as done in Section 1, be reversed?

Problem 2 Examine the relationships between domains and generalized ultrametric spaces.

Problem 3 To what extent can Theorem 2.5 be reversed?

Problem 4 Try to find a contructive proof of Theorem 1.4 in order to find a fixed point of the function given there in the hypothesis.

References

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