A uniform approach to logic programming semantics

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Abstract

Part of the theory of logic programming and nonymonotonic reasoning concerns the study of fixed-point semantics for these paradigms. Several different semantics have been proposed during the last two decades, and some have been more successful and acknowledged than others. The rationales behind those various semantics have been manifold, depending on one's point of view, which may be that of a programmer or inspired by commonsense reasoning, and consequently the constructions which lead to these semantics are technically very diverse, and the exact relationships between them have not yet been fully understood. In this paper, we present a conceptually new method, based on level mappings, which allows to provide uniform characterizations of different semantics for logic programs. We will display our approach by giving new and uniform characterizations of some of the major semantics, more particular of the least model semantics for definite programs, of the Fitting semantics, and of the well-founded semantics. A novel characterization of the weakly perfect model semantics will also be provided.

Contents

1 Introduction

Negation in logic programming differs from the negation of classical logic. Indeed, the quest for a satisfactory understanding of negation in logic programming is still inconclusive — although the issue has cooled down a bit recently — and has proved to be very stimulating for research activities in computational logic, and in particular amongst knowledge representation and reasoning researchers concerned with commonsense and nonmonotonic reasoning. During the last two decades, different interpretations of negation in logic programming have lead to the development of a variety of *declarative semantics*, as they are called. Some early research efforts for establishing a satisfactory declarative semantics for negation as failure and its variants, as featured by the resolution-based Prolog family of logic programming systems, have later on been merged with nonmonotonic frameworks for commonsense reasoning, culminating recently in the development of so-called answer set programming systems, like SMODELS or DLV [MT99, Lif02, SNS0x].

Systematically, one can understand Fitting's proposal [Fit85] of a *Kripke-Kleene semantics* — also known as *Fitting semantics* — as a cornerstone which plays a fundamental role both for resolution-based and nonmonotonic reasoning inspired logic programming. Indeed, his proposal, which is based on a monotonic semantic operator in Kleene's strong three-valued logic, has been pursued in both communities, for example by Kunen [Kun87] for giving a semantics for pure Prolog, and by Apt and Pedreschi [AP93] in their fundamental paper on termination analysis of negation as failure, leading to the notion of *acceptable program*. On the other hand, however, Fitting himself [Fit91a, Fit02], using a bilattice-based approach which was further developed by Denecker, Marek, and Truszczynski [DMT00], tied his semantics closely to the major semantics inspired by nonmonotonic reasoning, namely the *stable model semantics* due to Gelfond and Lifschitz [GL88], which is based on a nonmonotonic semantic operator, and the *well-founded semantics* due to van Gelder, Ross, and Schlipf [vGRS91], originally defined using a different monotonic operator in three-valued logic together with a notion of unfoundedness.

Another fundamental idea which was recognised in both communities was that of *stratification*, with the underlying idea of restricting attention to certain kinds of programs in which recursion through negation is prevented. Apt, Blair, and Walker [ABW88] proposed a variant of resolution suitable for these programs, while Przymusinski [Prz88] and van Gelder [vG88] generalized the notion to *local* stratification. Przymusinski [Prz88] developed the *perfect model semantics* for locally stratified programs, and together with Przymusinska [PP90] generalized it later to a three-valued setting as the *weakly perfect model semantics*.

The semantics mentioned so far are defined and characterized using a variety of different techniques and constructions, including monotonic and nonmonotonic semantic operators in two- and three-valued logics, program transformations, level mappings, restrictions to suitable subprograms, detection of cyclic dependencies etc. Relationships between the semantics have been established, but even a simple comparison of the respective models in restricted cases could be rather tedious. So, in this paper, we propose a methodology which allows to obtain uniform characterizations of all semantics previously mentioned, and we believe that it will scale up well to most semantics based on monotonic operators, and also to some nonmonotonic operators, and to extensions of the logic programming paradigm including disjunctive conclusions and uncertainty. The characterizations will allow immediate comparison between the semantics, and once obtained we will easily be able to make some new and interesting observations, including the fact that the well-founded semantics can formally be understood as a Fitting semantics augmented with a form of stratification. Indeed we will note that from this novel perspective the well-founded semantics captures the idea of stratification much better than the weakly perfect model semantics, thus providing a formal explanation for the historic fact that the latter has not received as much attention as the former.

The main tool which will be employed for our characterizations is the notion of *level mapping*. Level mappings are mappings from Herbrand bases to ordinals, i.e. they induce orderings on the set of all ground atoms while disallowing infinite descending chains. They have been a technical tool in a variety of contexts, including termination analysis for resolution-based logic programming as studied by Bezem [Bez89], Apt and Pedreschi [AP93], Marchiori [Mar96], Pedreschi, Ruggieri, and Smaus [PRS02], and others, where they appear naturally since ordinals are well-orderings. They have been used for defining classes of programs with desirable semantic properties, e.g. by Apt, Blair, and Walker [ABW88], Przymusinski [Prz88] and Cavedon [Cav91], and they are intertwined with topological investigations of fixed-point semantics in logic programming, as studied e.g. by Fitting [Fit94, Fit02], and by Hitzler and Seda [Sed95, Sed97, Hit01, HS0x]. Level mappings are also relevant to some aspects of the study of relationships between logic programming and artificial neural networks, as studied by Hölldobler, Kalinke, and Störr [HKS99] and by Hitzler and Seda [HS00]. In our novel approach to uniform characterizations of different semantics, we will use them as a technical tool for capturing dependencies between atoms in a program.

The paper is structured as follows. Section 2 contains preliminaries which are needed to make the paper relatively self-contained. The subsequent sections contain the announced uniform characterziations of the least model semantics for definite programs and the stable model semantics in Section 3, of the Fitting semantics in Section 4, of the well-founded semantics in Section 5, and of the weakly perfect model semantics in Section 6. Related work will be discussed in Section 7, and we close with conclusions and a discussion of further work in Section 8.

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2 Preliminaries and Notation

A (*normal*) *logic program* is a finite set of (universally quantified) *clauses* of the form $\forall(A \leftarrow A_1 \wedge \cdots \wedge A_n \wedge \neg B_1 \wedge \cdots \wedge \neg B_m)$, commonly written as $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$, where A, A_i, and B_j, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, are atoms over some given first order language. A is called the *head* of the clause, while the remaining atoms make up the *body* of the clause, and depending on context, a body of a clause will be a set of literals (i.e. atoms or negated atoms) or the conjunction of these literals. Care will be taken that this identification does not cause confusion. We allow a body, i.e. a conjunction, to be empty, in which case it always evaluates to true. A clause with empty body is called a *unit clause* or a *fact*. A clause is called *definite*, if it contains no negation symbol. A program is called *definite* if it consists only of definite clauses. We will usually denote atoms with A or B, and literals, which may be atoms or negated atoms, by L or K.

Given a logic program P, we can extract from it the components of a first order language. The corresponding set of ground atoms, i.e. the *Herbrand base* of the program, will be denoted by B_P. For a subset $I \subseteq B_P$, we set $\neg I = \{\neg A \mid A \in B_P\}$. The set of all ground instances of P with respect to B_P will be denoted by $\mathsf{ground}(P)$. For $I \subseteq B_P \cup \neg B_P$ we say that A is *true with respect to* (or *in*) I if $A \in I$, we say that A is *false with respect to* (or *in*) I if $\neg A \in I$, and if neither is the case, we say that A is *undefined with respect to* (or *in*) I. A (*three-valued* or *partial*) *interpretation* I for P is a subset of $B_P \cup \neg B_P$ which is *consistent*, i.e. whenever $A \in I$ then $\neg A \notin I$. A body, i.e. a conjunction of literals, is true in an interpretation I if every literal in the body is true in I, it is false in I if one of its literals is false in I, and otherwise it is undefined in I. For a negative literal $L = \neg A$ we will find it convenient to write $\neg L \in I$ if $A \in I$ and say that L is false in I etc. in this case. By I_P we denote the set of all (three-valued) interpretations of P. It is a complete partial order (cpo) via set-inclusion, i.e. it contains the empty set as least element, and every ascending chain has a supremum, namely its union. A *model* of P is an interpretation $I \in I_P$ such that for each clause $A \leftarrow \mathsf{body}$ we have that $\mathsf{body} \subseteq I$ implies $A \in I$. A *total* interpretation is an interpretation I such that no $A \in B_P$ is undefined in I.

For an interpretation I and a program P, an *I-partial level mapping* for P is a partial mapping $l : B_P \rightarrow \alpha$ with domain $\mathsf{dom}(l) = \{A \mid A \in I \text{ or } \neg A \in I\}$, where α is some (countable) ordinal. We extend every level mapping to literals by setting $l(\neg A) = l(A)$ for all $A \in \mathsf{dom}(l)$. A (*total*) *level mapping* is a total mapping $l : B_P \rightarrow \alpha$ for some (countable) ordinal α.

Given a normal logic program P and some $I \subseteq B_P \cup \neg B_P$, we say that $U \subseteq B_P$ is an *unfounded set* (*of* P) *with respect to* I if each atom $A \in U$ satisfies the following condition: For each clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ (at least) one of the following holds.

(Ui) Some (positive or negative) literal in body is false in I.

(Uii) Some (non-negated) atom in body occurs in U.

Given a normal logic program P, we define the following operators on $B_P \cup \neg B_P$. $T_P(I)$ is the set of all $A \in B_P$ such that there exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that body is true in I. $F_P(I)$ is the set of all $A \in B_P$ such that for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we have that body is false in I. Both T_P and F_P map elements of I_P to elements of I_P. Now define the operator $\Phi_P : I_P \to I_P$ by

$$\Phi_P(I) = T_P(I) \cup \neg F_P(I).$$

This operator is due to Fitting [Fit85] and is monotonic on the cpo I_P, hence has a least fixed point by the Tarski fixed-point theorem, and we can obtain this fixed point by defining, for each monotonic operator F, that $F\uparrow 0 = \emptyset$, $F\uparrow(\alpha+1) = F(F\uparrow\alpha)$ for any ordinal α, and $F\uparrow\beta = \bigcup_{\gamma<\beta} F\uparrow\gamma$ for any limit ordinal β, and the least fixed point $\mathsf{lfp}(F)$ of F is obtained as $F\uparrow\alpha$ for some ordinal α. The least fixed point of Φ_P is called the *Kripke-Kleene model* or *Fitting model* of P, determining the *Fitting semantics* of P.

2.1 Example Let P be the program consisting of the two clauses $p \leftarrow p$ and $q \leftarrow \neg r$. Then $\Phi_P\uparrow 1 = \{\neg r\}$, and $\Phi_P\uparrow 2 = \{q, \neg r\} = \Phi_P\uparrow 3$ is the Fitting model of P.

Now, for $I \subseteq B_P \cup \neg B_P$, let $U_P(I)$ be the greatest unfounded set (of P) with respect to I, which always exists due to van Gelder, Ross, and Schlipf [vGRS91]. Finally, define

$$W_P(I) = T_P(I) \cup \neg U_P(I)$$

for all $I \subseteq B_P \cup \neg B_P$. The operator W_P, which operates on the cpo $B_P \cup \neg B_P$, is due to van Gelder et al. [vGRS91] and is monotonic, hence has a least fixed point by the Tarski fixed-point theorem, as above for Φ_P. It turns out that $W_P\uparrow\alpha$ is in I_P for each ordinal α, and so the least fixed point of W_P is also in I_P and is called the *well-founded model* of P, giving the *well-founded semantics* of P.

2.2 Example Let P be the program consisting of the following clauses.

$$\begin{aligned} s &\leftarrow q \\ q &\leftarrow \neg p \\ p &\leftarrow p \\ r &\leftarrow \neg r \end{aligned}$$

Then $\{p\}$ is the largest unfounded set of P with respect to $\emptyset$ and we obtain

$$\begin{aligned} &W_P\uparrow 1 = \{\neg p\}, \\ &W_P\uparrow 2 = \{\neg p, q\}, \qquad \text{and} \\ &W_P\uparrow 3 = \{\neg p, q, s\} = W_P\uparrow 4. \end{aligned}$$

Given a program P, we define the operator T_P^+ on subsets of B_P by $T_P^+(I) = T_P(I \cup \neg(B_P \setminus I))$. It is well-known that for definite programs this operator is monotonic on the set of all subsets of B_P, with respect to subset inclusion. Indeed it is Scott-continuous [Llo88, SHLG94] and, via Kleene's fixed-point theorem, achieves its least fixed point M as the supremum of the iterates $T_P^+\uparrow n$ for $n \in \mathbb{N}$. So $M = \mathsf{lfp}(T_P^+) = T_P^+\uparrow\omega$ is *the least two-valued model* of P. In turn, we can identify M with the total interpretation $M \cup \neg(B_P \setminus M)$, which we will call the *definite* (*partial*) *model* of P.

2.3 Example Let P be the program consisting of the clauses

$$\begin{aligned} p(0) &\leftarrow \qquad \text{and} \\ p(s(X)) &\leftarrow p(X), \end{aligned}$$

where X denotes a variable and 0 a constant symbol. Write $s^n(0)$ for the term $s(\ldots s(0)\ldots)$ in which the symbol s appears n times. Then

$$T_P^+\uparrow n = \left\{p\left(s^k(0)\right) \mid k < n\right\}$$

for all $n \in \mathbb{N}$ and $\{p(s^n(0)) \mid n \in \mathbb{N}\}$ is the least two-valued model of P.

In order to avoid confusion, we will use the following terminology: the notion of *interpretation* will by default denote consistent subsets of $B_P \cup \neg B_P$, i.e. interpretations in three-valued logic. We will sometimes emphasize this point by using the notion *partial interpretation*. By *two-valued interpretations* we mean subsets of B_P. Given a partial interpretation I, we set $I^+ = I \cap B_P$ and $I^- = \{A \in B_P \mid \neg A \in I\}$. Each two-valued interpretation I can be identified with the partial interpretation $I' = I \cup \neg(B_P \setminus I)$. Both, interpretations and two-valued interpretations, are ordered by subset inclusion. We note however, that these two orderings differ: If $I \subseteq B_P$, for example, then I' is always a maximal element in the ordering for partial interpretations, while I is in general not maximal as a two-valued interpretation. The two orderings correspond to the knowledge- and the truth-ordering due to Fitting [Fit91a].

There is a semantics using two-valued logic, the stable model semantics due to Gelfond and Lifschitz [GL88], which is intimately related to the well-founded semantics. Let P be a normal program, and let $M \subseteq B_P$ be a set of atoms. Then we define P/M to be the (ground) program consisting of all clauses $A \leftarrow A_1, \ldots, A_n$ for which there is a clause $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $B_1, \ldots, B_m \not\in M$. Since P/M does no longer contain negation, it has a least two-valued model $T_{P/M}^+\uparrow\omega$. For any two-valued interpretation I we can therefore define the operator $\mathrm{GL}_P(I) = T_{P/I}^+\uparrow\omega$, and call M a *stable model* of the normal program P if it is a fixed point of the operator GL_P, i.e. if $M = \mathrm{GL}_P(M) = T_{P/M}^+\uparrow\omega$. As it turns out, the operator GL_P is in general not monotonic for normal programs P. However it is *antitonic*, i.e. whenever $I \subseteq J \subseteq B_P$ then $\mathrm{GL}_P(J) \subseteq \mathrm{GL}_P(I)$. As a consequence, the operator GL_P^2, obtained by applying GL_P twice, is monotonic and hence has a least fixed point L_P and a greatest fixed point G_P. Van Gelder [vG89] has shown that $\mathrm{GL}_P(L_P) = G_P$, $L_P = \mathrm{GL}_P(G_P)$, and that $L_P \cup \neg(B_P \setminus G_P)$ coincides with the well-founded model of P. This is called the *alternating fixed point characterization* of the well-founded semantics.

2.4 Example Consider the program P from Example 2.2. The subprogram Q consisting of the first three clauses of the program P has stable model $M = \{s, q\}$, which can be verified by noting that Q/M consists of the clauses

$$\begin{aligned} s &\leftarrow q, \\ q &\leftarrow \qquad \text{and} \\ p &\leftarrow p, \end{aligned}$$

and has M as its least two-valued model.

For the program P we obtain

$$\begin{aligned} \mathrm{GL}_P(\emptyset) &= \{q,s,r\}, \\ \mathrm{GL}_P(\{q,s,r\}) &= \{q,s\} = \mathrm{GL}_P^2(\{q,s\}), \qquad \text{and} \\ \mathrm{GL}_P(B_P) &= \emptyset. \end{aligned}$$

So $L_P = \{q,s\}$ while $G_P = \{q,s,r\}$, and $L_P \cup \neg(B_P \setminus G_P) = \{q,s,\neg p\}$ is the well-founded model of P.

3 Least and Stable Model Semantics

The most fundamental semantics in logic programming is based on the fact mentioned above that the operator T_P^+ has a least fixed point $M = T_P^+{\uparrow}\omega$ whenever P is definite. The two-valued interpretation M turns out to be the least two-valued model of the program, and is therefore canonically the model which should be considered for definite programs. Our first result characterizes the least model using level mappings, and serves to convey the main ideas underlying our method. It is a straightforward result but has, to the best of our knowledge, not been noted before.

3.1 Theorem Let P be a definite. Then there is a unique two-valued model M of P for which there exists a (total) level mapping $l : B_P \to \alpha$ such that for each atom $A \in M$ there exists a clause $A \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(P)$ with $A_i \in M$ and $l(A) > l(A_i)$ for all $i = 1, \ldots, n$. Furthermore, M is the least two-valued model of P.

Proof: Let M be the least two-valued model $T_P^+{\uparrow}\omega$, choose $\alpha = \omega$, and define $l : B_P \to \alpha$ by setting $l(A) = \mathsf{min}\{n \mid A \in T_P^+{\uparrow}(n+1)\}$, if $A \in M$, and by setting $l(A) = 0$, if $A \notin M$. From the fact that $\emptyset \subseteq T_P^+{\uparrow}1 \subseteq \ldots \subseteq T_P^+{\uparrow}n \subseteq \ldots \subseteq T_P^+{\uparrow}\omega = \bigcup_m T_P^+{\uparrow}m$, for each n, we see that l is well-defined and that the least model $T_P^+{\uparrow}\omega$ for P has the desired properties.

Conversely, if M is a two-valued model for P which satisfies the given condition for some mapping $l : B_P \to \alpha$, then it is easy to show, by induction on $l(A)$, that $A \in M$ implies $A \in T_P^+{\uparrow}(l(A)+1)$. This yields that $M \subseteq T_P^+{\uparrow}\omega$, and hence that $M = T_P^+{\uparrow}\omega$ by minimality of the model $T_P^+{\uparrow}\omega$. ∎

3.2 Example For the program P from Example 2.3 we obtain $l(p(s^n(0))) = n$ for the level mapping l defined in the proof of Theorem 3.1.

The proof of Theorem 3.1 can serve as a blueprint for obtaining characterizations if the semantics under consideration is based on the least fixed point of a monotonic operator F, and indeed our results for the Fitting semantics and the well-founded semantics, Theorems 4.2 and 5.2, together with their proofs, follow this scheme. In one direction, levels are assigned to atoms A according to the least ordinal α such that A is not undefined in $F \uparrow (\alpha + 1)$, and dependencies between atoms of some level and atoms of lower levels are captured by the nature of the considered operator, which will certainly vary from case to case. In Theorem 3.1, the condition thus obtained suffices for uniquely determining the least model, whereas in other cases which we will study later, so for the Fitting semantics and the well-founded semantics, the level mapping conditions will not suffice for unique characterization of the desired model. However, the desired model will in each case turn out to be the greatest among all models satisfying the given conditions. So in these cases it will remain to show, by transfinite induction on the level of some given atom A, that the truth value assigned to A by any model satisfying the given conditions is also assigned to A by $F\uparrow(l(A)+1)$, which at the same time proves that $\mathsf{lfp}(F)$ is the greatest model satisfying the given conditions. For the proof of Theorem 3.1, the proof method just described can be applied straightforwardly, however for more sophisticated operators may become technically challenging on the detailed level.

We now turn to the stable model semantics, which in the case of programs with negation has come to be the major semantics based on two-valued logic. The following characterization is in the spirit of our proposal, and is due to Fages [Fag94]. It is striking in its similarity to the characterization of the least model for definite programs in Theorem 3.1. For completeness of our exhibition, we include a proof of the fact.

3.3 Theorem Let P be normal. Then a two-valued model $M \subseteq B_P$ of P is a stable model of P if and only if there exists a (total) level mapping $l : B_P \to \alpha$ such that for each $A \in M$ there exists $A \leftarrow A_1, \dots, A_n \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(P)$ with $A_i \in M$, $B_j \notin M$, and $l(A) > l(A_i)$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

Proof: Let M be a stable model of P, i.e. $\mathrm{GL}_P(M) = T^+_{P/M}\uparrow\omega = M$. Then M is the least model for P/M, hence is also a model for P, and, by Theorem 3.1, satisfies the required condition with respect to any level mapping l with $l(A) = \mathsf{min}\{n \mid A \in T_{P/M}\uparrow(n+1)\}$ for each $A \in M$. Conversely, let M be a model which satisfies the condition in the statement of the theorem. Then, for every $A \in M$, there is a clause C in $\mathsf{ground}(P)$ of the form $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_k$ such that the body of C is true in M and satisfies $l(A) > l(A_i)$ for all $i = 1, \dots, n$. But then, for every $A \in M$, there is a clause $A \leftarrow A_1, \dots, A_n$ in P/M whose body is true in M and such that $l(A) > l(A_i)$ for all $i = 1, \dots, n$. By Theorem 3.1, this means that M is the least model for P/M, that is, $M = T^+_{P/M}\uparrow\omega = \mathrm{GL}(M)$. ■

The proof of Theorem 3.3 just given partly follows the proof scheme discussed previously, by considering the monotonic operator $T^+_{P/M}$, which is used for defining stable models.

3.4 Example Recall the program P from Example 2.2, and consider the program Q consisting of the first three clauses of P. We already noted in Example 2.4 that Q has stable model $\{s, q\}$. A corresponding level mapping, as defined in the proof of Theorem 3.3, satisfies $l(q) = 0$ and $l(s) = 1$, while $l(p)$ can be an arbitrary value.

4 Fitting Semantics

We next turn to the Fitting semantics. Following the proof scheme which we described in Section 3, we expect levels $l(A)$ to be assigned to atoms A such that $l(A)$ is the least α such that A is not undefined in $\Phi_P\uparrow(\alpha+1)$. An analysis of the operator Φ_P eventually yields the following conditions.

4.1 Definition Let P be a normal logic program, I be a model of P, and l be an I-partial level mapping for P. We say that P *satisfies* (F) *with respect to* I *and* l, if each $A \in \mathsf{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in I$ and there exists a clause $A \leftarrow L_1, \dots, L_n$ in $\mathsf{ground}(P)$ with $L_i \in I$ and $l(A) > l(L_i)$ for all i.

(Fii) $\neg A \in I$ and for each clause $A \leftarrow L_1, \dots, L_n$ in $\mathsf{ground}(P)$ there exists i with $\neg L_i \in I$ and $l(A) > l(L_i)$.

If $A \in \mathsf{dom}(l)$ satisfies (Fi), then we say that A *satisfies* (Fi) *with respect to* I *and* l, and similarly if $A \in \mathsf{dom}(l)$ satisfies (Fii).

We note that condition (Fi) is stronger than the condition used for characterizing stable models in Theorem 3.3. The proof of the next theorem closely follows our proof scheme.

4.2 Theorem Let P be a normal logic program with Fitting model M. Then M is the greatest model among all models I, for which there exists an I-partial level mapping l for P such that P satisfies (F) with respect to I and l.

Proof: Let M_P be the Fitting model of P and define the M_P-partial level mapping l_P as follows: $l_P(A) = \alpha$, where α is the least ordinal such that A is not undefined in $\Phi_P \uparrow (\alpha + 1)$. The proof will be established by showing the following facts: (1) P satisfies (F) with respect to M_P and l_P. (2) If I is a model of P and l an I-partial level mapping such that P satisfies (F) with respect to I and l, then $I \subseteq M_P$.

(1) Let $A \in \mathsf{dom}(l_P)$ and $l_P(A) = \alpha$. We consider two cases.

(Case i) If $A \in M_P$, then $A \in T_P(\Phi_P \uparrow \alpha)$, hence there exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that body is true in $\Phi_P \uparrow \alpha$. Thus, for all $L_i \in \mathsf{body}$ we have that $L_i \in \Phi_P \uparrow \alpha$, and hence $l_P(L_i) < \alpha$ and $L_i \in M_P$ for all i. Consequently, A satisfies (Fi) with respect to M_P and l_P.

(Case ii) If $\neg A \in M_P$, then $A \in F_P(\Phi_P \uparrow \alpha)$, hence for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ there exists $L \in \mathsf{body}$ with $\neg L \in \Phi_P \uparrow \alpha$ and $l_P(L) < \alpha$, hence $\neg L \in M_P$. Consequently, A satisfies (Fii) with respect to M_P and l_P, and we have established that fact (1) holds.

(2) We show via transfinite induction on $\alpha = l(A)$, that whenever $A \in I$ (respectively, $\neg A \in I$), then $A \in \Phi_P \uparrow (\alpha + 1)$ (respectively, $\neg A \in \Phi_P \uparrow (\alpha + 1)$). For the base case, note that if $l(A) = 0$, then $A \in I$ implies that A occurs as the head of a fact in $\mathsf{ground}(P)$, hence $A \in \Phi_P \uparrow 1$, and $\neg A \in I$ implies that there is no clause with head A in $\mathsf{ground}(P)$, hence $\neg A \in \Phi_P \uparrow 1$. So assume now that the induction hypothesis holds for all $B \in B_P$ with $l(B) < \alpha$. We consider two cases.

(Case i) If $A \in I$, then it satisfies (Fi) with respect to I and l. Hence there is a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that $\mathsf{body} \subseteq I$ and $l(K) < \alpha$ for all $K \in \mathsf{body}$. Hence $\mathsf{body} \subseteq M_P$ by induction hypothesis, and since M_P is a model of P we obtain $A \in M_P$.

(Case ii) If $\neg A \in I$, then A satisfies (Fii) with respect to I and l. Hence for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we have that there is $K \in \mathsf{body}$ with $\neg K \in I$ and $l(K) < \alpha$. Hence for all these K we have $\neg K \in M_P$ by induction hypothesis, and consequently for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we obtain that body is false in M_P. Since $M_P = \Phi_P(M_P)$ is a fixed point of the Φ_P-operator, we obtain $\neg A \in M_P$. This establishes fact (2) and concludes the proof. ■

4.3 Example Consider the program P from Example 2.1. Then the level mapping l, as defined in the proof of Theorem 4.2, satsifies $l(r) = 0$ and $l(q) = 1$.

It is interesting to consider the special case where the Fitting model is total. Programs with this property are called Φ-*accessible* [HS99, HS0x], and include e.g. the acceptable programs due to Apt and Pedreschi [AP93].

4.4 Corollary A normal logic program P has a total Fitting model if and only if there is a total model I of P and a (total) level mapping l for P such that P satisfies (F) with respect to I and l.

The result follows immediately as a special case of Theorem 4.2, and is closely related to results reported in [HS99, HS0x]. The reader familiar with acceptable programs will also note the close relationship between Corollary 4.4 and the defining conditions for acceptable programs. Indeed, the theorem due to Apt and Pedreschi [AP93], which says that every acceptable program has a total Fitting model, follows without any effort from our result. It also follows immediately, by comparing Corollary 4.4 and Theorem 3.3, that a total Fitting model is always stable, which is a well-known fact.

5 Well-Founded Semantics

The characterization of the well-founded model again closely follows our proof scheme. Before discussing this, though, we will take a short detour which will eventually reveal a surprising fact about the well-founded semantics: From our new perspective the well-founded semantics can be understood as a stratified version of the Fitting semantics.

Let us first recall the definition of a (locally) stratified program, due to Apt, Blair, Walker, and Przymusinski [ABW88, Prz88]: A normal logic program is called *locally stratified* if there exists a (total) level mapping $l : B_P \to \alpha$, for some ordinal α, such that for each clause $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ we have that $l(A) \geq l(A_i)$ and $l(A) > l(B_j)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

The notion of (locally) stratifed program, as already mentioned in the introduction, was developed with the idea of *preventing recursion through negation*, while allowing recursion through positive dependencies. There exist locally stratified programs which do not have a total Fitting model and vice versa. Indeed, the program consisting of the single clause $p \leftarrow p$ is locally stratified but p remains undefined in the Fitting model. Conversely, the program consisting of the two clauses $q \leftarrow$ and $q \leftarrow \neg q$ is not locally stratified but its Fitting model assigns to q the truth value *true*.

By comparing Definition 4.1 with the definition of locally stratified programs, we notice that condition (Fii) requires a strict decrease of level between the head and a literal in the rule, independent of this literal being positive or negative. But, on the other hand, condition (Fii) imposes no further restrictions on the remaining body literals, while the notion of local stratification does. These considerations motivate the substitution of condition (Fii) by the condition (WFii), as given in the following definition.

5.1 Definition Let P be a normal logic program, I be a model of P, and l be an I-partial level mapping for P. We say that P *satisfies* (WF) *with respect to* I *and* l, if each $A \in \mathsf{dom}(l)$ satisfies one of the following conditions.

(WFi) $A \in I$ and there exists a clause $A \leftarrow L_1, \dots, L_n$ in $\mathsf{ground}(P)$ with $L_i \in I$ and $l(A) > l(L_i)$ for all i.

(WFii) $\neg A \in I$ and for each clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(P)$ (at least) one of the following conditions holds:

(WFiia) There exists $i \in \{1, \dots, n\}$ with $\neg A_i \in I$ and $l(A) \geq l(A_i)$.

(WFiib) There exists $j \in \{1, \dots, m\}$ with $B_j \in I$ and $l(A) > l(B_j)$.

If $A \in \mathsf{dom}(l)$ satisfies (WFi), then we say that A *satisfies* (WFi) *with respect to* I *and* l, and similarly if $A \in \mathsf{dom}(l)$ satisfies (WFii).

We note that conditions (Fi) and (WFi) are identical. Indeed, replacing (WFi) by a stratified version such as the following seems not satisfactory.

(SFi) $A \in I$ and there exists a clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(P)$ with $A_i, B_j \in I$, $l(A) \geq l(A_i)$, and $l(A) > l(B_j)$ for all i and j.

If we replace condition (WFi) by condition (SFi), then it is not guaranteed that for any given program there is a greatest model satisfying the desired properties: Consider the program consisting of the two clauses $p \leftarrow p$ and $q \leftarrow \neg p$, and the two (total) models $\{p, \neg q\}$ and $\{\neg p, q\}$, which are incomparable, and the level mapping l with $l(p) = 0$ and $l(q) = 1$.

So, in the light of Theorem 4.2, Definition 5.1 should provide a natural "stratified version" of the Fitting semantics. And indeed it does, and furthermore, the resulting semantics coincides with the well-founded semantics, which is a very satisfactory result. The proof of the fact again follows our proof scheme, but is slightly more involved due to the necessary treatment of unfounded sets.

5.2 Theorem Let P be a normal logic program with well-founded model M. Then M is the greatest model among all models I, for which there exists an I-partial level mapping l for P such that P satisfies (WF) with respect to I and l.

Proof: Let M_P be the well-founded model of P and define the M_P-partial level mapping l_P as follows: $l_P(A) = \alpha$, where α is the least ordinal such that A is not undefined in $W_P \uparrow (\alpha + 1)$. The proof will be established by showing the following facts: (1) P satisfies (WF) with respect to M_P and l_P. (2) If I is a model of P and l an I-partial level mapping such that P satisfies (WF) with respect to I and l, then $I \subseteq M_P$.

(1) Let $A \in \mathsf{dom}(l_P)$ and $l_P(A) = \alpha$. We consider two cases.

(Case i) If $A \in M_P$, then $A \in T_P(W_P \uparrow \alpha)$, hence there exists a clause $A \leftarrow \mathtt{body}$ in $\mathsf{ground}(P)$ such that $\mathtt{body}$ is true in $W_P \uparrow \alpha$. Thus, for all $L_i \in \mathtt{body}$ we have that $L_i \in W_P \uparrow \alpha$.

Hence, $l_P(L_i) < \alpha$ and $L_i \in M_P$ for all i. Consequently, A satisfies (WFi) with respect to M_P and l_P.

(Case ii) If $\neg A \in M_P$, then $A \in U_P(W_P \uparrow \alpha)$, i.e. A is contained in the greatest unfounded set of P with respect to $W_P \uparrow \alpha$. Hence for each clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$, at least one of (Ui) or (Uii) holds for this clause with respect to $W_P \uparrow \alpha$ and the unfounded set $U_P(W_P \uparrow \alpha)$. If (Ui) holds, then there exists some literal $L \in \mathsf{body}$ with $\neg L \in W_P \uparrow \alpha$. Hence $l_P(L) < \alpha$ and condition (WFiia) holds relative to M_P and l_P if L is an atom, or condition (WFiib) holds relative to M_P and l_P if L is a negated atom. On the other hand, if (Uii) holds, then some (non-negated) atom B in body occurs in $U_P(W_P \uparrow \alpha)$. Hence $l_P(B) \leq l_P(A)$ and A satisfies (WFiia) with respect to M_P and l_P. Thus we have established that fact (1) holds.

(2) We show via transfinite induction on $\alpha = l(A)$, that whenever $A \in I$ (respectively, $\neg A \in I$), then $A \in W_P \uparrow (\alpha + 1)$ (respectively, $\neg A \in W_P \uparrow (\alpha + 1)$). For the base case, note that if $l(A) = 0$, then $A \in I$ implies that A occurs as the head of a fact in $\mathsf{ground}(P)$. Hence, $A \in W_P \uparrow 1$. If $\neg A \in I$, then consider the set U of all atoms B with $l(B) = 0$ and $\neg B \in I$. We show that U is an unfounded set of P with respect to $W_P \uparrow 0$, and this suffices since it implies $\neg A \in W_P \uparrow 1$ by the fact that $A \in U$. So let $C \in U$ and let $C \leftarrow \mathsf{body}$ be a clause in $\mathsf{ground}(P)$. Since $\neg C \in I$, and $l(C) = 0$, we have that C satisfies (WFiia) with respect to I and l, and so condition (Uii) is satisfied showing that U is an unfounded set of P with respect to I. Assume now that the induction hypothesis holds for all $B \in B_P$ with $l(B) < \alpha$. We consider two cases.

(Case i) If $A \in I$, then it satisfies (WFi) with respect to I and l. Hence there is a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that $\mathsf{body} \subseteq I$ and $l(K) < \alpha$ for all $K \in \mathsf{body}$. Hence $\mathsf{body} \subseteq W_P \uparrow \alpha$, and we obtain $A \in T_P(W_P \uparrow \alpha)$ as required.

(Case ii) If $\neg A \in I$, consider the set U of all atoms B with $l(B) = \alpha$ and $\neg B \in I$. We show that U is an unfounded set of P with respect to $W_P \uparrow \alpha$, and this suffices since it implies $\neg A \in W_P \uparrow (\alpha + 1)$ by the fact that $A \in U$. So let $C \in U$ and let $C \leftarrow \mathsf{body}$ be a clause in $\mathsf{ground}(P)$. Since $\neg C \in I$, we have that C satisfies (WFii) with respect to I and l. If there is a literal $L \in \mathsf{body}$ with $\neg L \in I$ and $l(L) < l(C)$, then by the induction hypothesis we obtain $\neg L \in W_P \uparrow \alpha$, so condition (Ui) is satisfied for the clause $C \leftarrow \mathsf{body}$ with respect to $W_P \uparrow \alpha$ and U. In the remaining case we have that C satisfies condition (WFiia), and there exists an atom $B \in \mathsf{body}$ with $\neg B \in I$ and $l(B) = l(C)$. Hence, $B \in U$ showing that condition (Uii) is satisfied for the clause $C \leftarrow \mathsf{body}$ with respect to $W_P \uparrow \alpha$ and U. Hence U is an unfounded set of P with respect to $W_P \uparrow \alpha$. ■

5.3 Example Consider the program P from Example 2.2. With notation from the proof of Theorem 5.2, we obtain $l(p) = 0$, $l(q) = 1$, and $l(s) = 2$.

As a special case, we consider programs with total well-founded model. The following corollary follows without effort from Theorem 5.2.

5.4 Corollary A normal logic program P has a total well-founded model if and only if there is a total model I of P and a (total) level mapping l such that P satisfies (WF) with respect to I and l.

As a further example for the application of our proof scheme, we use Theorem 5.2 in order to prove a result by van Gelder [vG89] which we mentioned in the introduction, concerning the alternating fixed-point characterization of the well-founded semantics. Let us first introduce some temporary notation, where P is an arbitrary program.

$$\begin{aligned}
L_0 &= \emptyset \\
G_0 &= B_P \\
L_{\alpha+1} &= \mathrm{GL}_P(G_\alpha) && \text{for any ordinal } \alpha \\
G_{\alpha+1} &= \mathrm{GL}_P(L_\alpha) && \text{for any ordinal } \alpha \\
L_\alpha &= \bigcup_{\beta<\alpha} L_\beta && \text{for limit ordinal } \alpha \\
G_\alpha &= \bigcap_{\beta<\alpha} G_\beta && \text{for limit ordinal } \alpha
\end{aligned}$$

Since $\emptyset \subseteq B_P$, we obtain $L_0 \subseteq L_1 \subseteq G_1 \subseteq G_0$ and, by transfinite induction, it can easily be shown that $L_\alpha \subseteq L_\beta \subseteq G_\beta \subseteq G_\alpha$ whenever $\alpha \leq \beta$. In order to apply our proof scheme, we need to detect a monotonic operator, or at least some kind of monotonic construction, underlying the alternative fixed-point characterization. The assignment $(L_\alpha, G_\alpha) \mapsto (L_{\alpha+1}, G_{\alpha+1})$, using the temporary notation introduced above, will serve for this purpose. The proof of the following theorem is based on it and our general proof scheme, with modifications where necessary, for example for accomodating the fact that $G_{\alpha+1}$ is not defined using G_α, but rather L_α, and that we work with the complements $B_P \setminus G_\alpha$ instead of the sets G_α.

5.5 Theorem Let P be a normal program. Then $M = L_P \cup \neg(B_P \setminus G_P)$ is the well-founded model of P.

Proof: First, we define an M-partial level mapping l. For convenience, we will take as image set of l, pairs (α, n) of ordinals, where $n \leq \omega$, with the lexicographic ordering. This can be done without loss of generality because any set of pairs of ordinals, lexicographically ordered, is certainly well-ordered and therefore order-isomorphic to an ordinal. For $A \in L_P$, let $l(A)$ be the pair (α, n), where α is the least ordinal such that $A \in L_{\alpha+1}$, and n is the least ordinal such that $A \in T_{P/G_\alpha}\uparrow(n+1)$. For $B \notin G_P$, let $l(B)$ be the pair (β, ω), where β is the least ordinal such that $B \notin G_{\beta+1}$. We show next by transfinite induction that P satisfies (WF) with respect to M and l.

Let $A \in L_1 = T_{P/B_P}\uparrow\omega$. Since P/B_P consists of exactly all clauses from $\mathsf{ground}(P)$ which contain no negation, we have that A is contained in the least two-valued model for a definite subprogram of P, namely P/B_P, and (WFi) is satisfied by Theorem 3.1. Now let $\neg B \in \neg(B_P \setminus G_P)$ be such that $B \in (B_P \setminus G_1) = B_P \setminus T_{P/\emptyset}\uparrow\omega$. Since $P/\emptyset$ contains all clauses from $\mathsf{ground}(P)$ with all negative literals removed, we obtain that each clause in $\mathsf{ground}(P)$ with head B must contain a positive body literal $C \notin G_1$, which, by definition of l, must have the same level as B, hence (WFiia) is satisfied.

Assume now that, for some ordinal α, we have shown that A satisfies (WF) with respect to M and l for all $n \leq \omega$ and all $A \in B_P$ with $l(A) \leq (\alpha, n)$.

Let $A \in L_{\alpha+1} \setminus L_\alpha = T_{P/G_\alpha}\uparrow\omega \setminus L_\alpha$. Then $A \in T_{P/G_\alpha}\uparrow n \setminus L_\alpha$ for some $n \in \mathbb{N}$; note that all (negative) literals which were removed by the Gelfond-Lifschitz transformation from clauses with head A have level less than $(\alpha, 0)$. Then the assertion that A satisfies (WF) with respect to M and l follows again by Theorem 3.1.

Let $A \in (B_P \setminus G_{\alpha+1}) \cap G_\alpha$. Then $A \notin T_{P/L_\alpha}{\uparrow}\omega$. Let $A \leftarrow A_1, \dots, A_k, \neg B_1, \dots, \neg B_m$ be a clause in $\mathsf{ground}(P)$. If $B_j \in L_\alpha$ for some j, then $l(A) > l(B_j)$. Otherwise, since $A \notin T_{P/L_\alpha}{\uparrow}\omega$, we have that there exists A_i with $A_i \notin T_{P/L_\alpha}{\uparrow}\omega$, and hence $l(A) \geq l(A_i)$, and this suffices.

This finishes the proof that P satisfies (WF) with respect to M and l. It therefore only remains to show that M is greatest with this property.

So assume that $M_1 \neq M$ is the greatest model such that P satisfies (WF) with respect to M_1 and some M_1-partial level mapping l_1.

Assume $L \in M_1 \setminus M$ and, without loss of generality, let the literal L be chosen such that $l_1(L)$ is minimal. We consider the following two cases.

(Case i) If $L = A$ is an atom, then there exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that $l_1(L) < l_1(A)$ for all literals L in body, and such that body is true in M_1. Hence, body is true in M and $A \leftarrow \mathsf{body}$ transforms to a clause $A \leftarrow A_1, \dots, A_n$ in P/G_P with $A_1, \dots, A_n \in L_P = T_{P/G_P}{\uparrow}\omega$. But this implies $A \in M$, contradicting $A \in M_1 \setminus M$.

(Case ii) If $L = \neg A \in M_1 \setminus M$ is a negated atom, then $\neg A \in M_1$ and $A \in G_P = T_{P/L_P}{\uparrow}\omega$, so $A \in T_{P/L_P}{\uparrow}n$ for some $n \in \mathbb{N}$. We show by induction on n that this leads to a contradiction, to finish the proof.

If $A \in T_{P/L_P}{\uparrow}1$, then there is a unit clause $A \leftarrow$ in P/L_P, and any corresponding clause $A \leftarrow \neg B_1, \dots, \neg B_k$ in $\mathsf{ground}(P)$ satisfies $B_1, \dots, B_k \notin L_P$. Since $\neg A \in M_1$, we also obtain by Theorem 5.2 that there is $i \in \{1, \dots, k\}$ such that $B_i \in M_1$ and $l_1(B_i) < l_1(A)$. By minimality of $l_1(A)$, we obtain $B_i \in M$, and hence $B_i \in L_P$, which contradicts $B_i \notin L_P$.

Now assume that there is no $\neg B \in M_1 \setminus M$ with $B \in T_{P/L_P}{\uparrow}k$ for any $k < n+1$, and let $\neg A \in M_1 \setminus M$ with $A \in T_{P/L_P}{\uparrow}(n+1)$. Then there is a clause $A \leftarrow A_1, \dots, A_m$ in P/L_P with $A_1, \dots, A_m \in T_{P/L_P}{\uparrow}n \subseteq G_P$, and we note that we cannot have $\neg A_i \in M_1 \setminus M$ for any $i \in \{1, \dots, m\}$, by our current induction hypothesis. Furthermore, it is also impossible for $\neg A_i$ to belong to M for any i, otherwise we would have $A_i \in B_P \setminus G_P$. Thus, we conclude that we cannot have $\neg A_i \in M_1$ for any i. Moreover, there is a corresponding clause $A \leftarrow A_1, \dots, A_m, \neg B_1, \dots, \neg B_{m_1}$ in $\mathsf{ground}(P)$ with $B_1, \dots, B_{m_1} \notin L_P$. Hence, by Theorem 5.2, we know that there is $i \in \{1, \dots, m_1\}$ such that $B_i \in M_1$ and $l_1(B_i) < l_1(A)$. By minimality of $l_1(A)$, we conclude that $B_i \in M$, so that $B_i \in L_P$, and this contradicts $B_i \notin L_P$. ■

5.6 Example Consider again the program P from Examples 2.2, 2.4, and 5.3. With notation from the proof of Theorem 5.5 we get $l(q) = (1, 0)$, $l(s) = (1, 1)$, and $l(p) = (0, \omega)$.

6 Weakly Perfect Model Semantics

By applying our proof scheme, we have obtained new and uniform characterizations of the Fitting semantics and the well-founded semantics, and argued that the well-founded semantics is a stratified version of the Fitting semantics. Our argumentation is based on the key intuition underlying the notion of stratification, that recursion should be allowed through positive dependencies, but be forbidden through negative dependencies. As we have seen in Theorem 5.2, the well-founded semantics provides this for a setting in three-valued logic. Historically, a different semantics, given by the so-called weakly perfect model associated with each program, was proposed by Przymusinska and Przymusinski [PP90] in order to carry over the intuition underlying the notion of stratification to a three-valued setting. In the following, we will characterize weakly perfect models via level mappings, in the spirit of our approach. We will thus have obtained uniform characterizations of the Fitting semantics, the well-founded semantics, and the weakly perfect model semantics, which makes it possible to easily compare them.

6.1 Definition Let P be a normal logic program, I be a model of P and l be an I-partial level mapping for P. We say that P *satisfies* (WS) *with respect to* I *and* l, if each $A \in \mathsf{dom}(l)$ satisfies one of the following conditions.

(WSi) $A \in I$ and there exists a clause $A \leftarrow L_1, \dots, L_n \in \mathsf{ground}(P)$ such that $L_i \in I$ and $l(A) > l(L_i)$ for all $i = 1, \dots, n$.

(WSii) $\neg A \in I$ and for each clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m \in \mathsf{ground}(P)$ (at least) one of the following three conditions holds.

- (WSiia) There exists i such that $\neg A_i \in I$ and $l(A) > l(A_i)$.
- (WSiib) For all k we have $l(A) \geq l(A_k)$, for all j we have $l(A) > l(B_j)$, and there exists i with $\neg A_i \in I$.
- (WSiic) There exists j such that $B_j \in I$ and $l(A) > l(B_j)$.

We observe that the condition (WSii) in the above theorem is more general than (Fii), and more restrictive than (WFii).

We will see below in Theorem 6.4, that Definition 6.1 captures the weakly perfect model, in the same way in which Definitions 4.1 and 5.1 capture the Fitting model, respectively the well-founded model.

In order to proceed with this, we first need to recall the definition of weakly perfect models due to Przymusinska and Przymusinski [PP90], and we will do this next. For ease of notation, it will be convenient to consider (countably infinite) propositional programs instead of programs over a first-order language. This is both common practice and no restriction, because the ground instantiation $\mathsf{ground}(P)$ of a given program P can be understood as a propositional program which may consist of a countably infinite number of clauses. Let us remark that our definition below differs slightly from the original one, and we will return to this point later. It nevertheless leads to exactly the same notion of weakly stratified program.

Let P be a (countably infinite propositional) normal logic program. An atom $A \in B_P$ *refers to* an atom $B \in B_P$ if B or $\neg B$ occurs as a body literal in a clause $A \leftarrow \mathsf{body}$ in P. A *refers negatively to* B if $\neg B$ occurs as a body literal in such a clause. We say that A *depends on* B if the pair (A, B) is in the transitive closure of the relation *refers to*, and we write this as $B \leq A$. We say that A *depends negatively on* B if there are $C, D \in B_P$ such that C refers negatively to D and the following hold: (1) $C \leq A$ or $C = A$ (the latter meaning identity). (2) $B \leq D$ or $B = D$. We write $B < A$ in this case. For $A, B \in B_P$, we write $A \sim B$ if either $A = B$, or A and B depend negatively on each other, i.e. if $A < B$ and $B < A$ both hold. The relation $\sim$ is an equivalence relation and its equivalence classes are called *components* of P. A component is *trivial* if it consists of a single element A with $A \not< A$.

Let C_1 and C_2 be two components of a program P. We write $C_1 \prec C_2$ if and only if $C_1 \neq C_2$ and for all $A_1 \in C_1$ there is $A_2 \in C_2$ with $A_1 < A_2$. A component C_1 is called *minimal* if there is no component C_2 with $C_2 \prec C_1$.

Given a normal logic program P, the *bottom stratum* $S(P)$ of P is the union of all minimal components of P. The *bottom layer* of P is the subprogram $L(P)$ of P which consists of all clauses from P with heads belonging to $S(P)$.

Given a (partial) interpretation I of P, we define the *reduct of P with respect to I* as the program P/I obtained from P by performing the following reductions. (1) Remove from P all clauses which contain a body literal L such that $\neg L \in I$ or whose head belongs to I. (2) Remove from all remaining clauses all body literals L with $L \in I$. (3) Remove from the resulting program all non-unit clauses, whose heads appear also as unit clauses in the program.

6.2 Definition The *weakly perfect model* M_P of a program P is defined by transfinite induction as follows. Let $P_0 = P$ and $M_0 = \emptyset$. For each (countable) ordinal $\alpha > 0$ such that programs P_δ and partial interpretations M_δ have already been defined for all $\delta < \alpha$, let

$$\begin{aligned}
&N_\alpha = \bigcup_{0<\delta<\alpha} M_\delta, \\
&P_\alpha = P/N_\alpha, \\
&R_\alpha \text{ is the set of all atoms which are undefined in } N_\alpha \\
&\qquad \text{and were eliminated from } P \text{ by reducing it with respect to } N_\alpha, \\
&S_\alpha = S(P_\alpha), \text{ and} \\
&L_\alpha = L(P_\alpha).
\end{aligned}$$

The construction then proceeds with one of the following three cases. (1) If P_α is empty, then the construction stops and $M_P = N_\alpha \cup \neg R_\alpha$ is the (*total*) *weakly perfect model* of P. (2) If the bottom stratum S_α is empty or if the bottom layer L_α contains a negative literal, then the construction also stops and $M_P = N_\alpha \cup \neg R_\alpha$ is the (*partial*) *weakly perfect model* of P. (3) In the remaining case L_α is a definite program, and we define $M_\alpha = H \cup \neg R_\alpha$, where H is the definite (partial) model of L_α, and the construction continues.

For every α, the set $S_\alpha \cup R_\alpha$ is called the *α-th stratum* of P and the program L_α is called the *α-th layer* of P.

A *weakly stratified program* is a program with a total weakly perfect model. The set of its strata is then called its *weak stratification.*

6.3 Example Consider the program P which consists of the following six clauses.

$$\begin{aligned}
a &\leftarrow \neg b \\
b &\leftarrow c, \neg a \\
b &\leftarrow c, \neg d \\
c &\leftarrow b, \neg e \\
d &\leftarrow e \\
e &\leftarrow d
\end{aligned}$$

Then $N_1 = M_1 = \{\neg d, \neg e\}$ and P/N_1 consists of the clauses

$$\begin{aligned} a &\leftarrow \neg b \\ b &\leftarrow c, \neg a \\ b &\leftarrow c \\ c &\leftarrow b. \end{aligned}$$

Its least component is $\{a, b, c\}$. The corresponding bottom layer, which is all of P/N_1, contains a negative literal, so the construction stops and $M_2 = N_1 = \{\neg d, \neg e\}$ is the (partial) weakly perfect model of P.

Let us return to the remark made earlier that our definition of weakly perfect model, as given in Definition 6.2, differs slightly from the version introduced by Przymusinska and Przymusinski [PP90]. In order to obtain the original definition, points (2) and (3) of Definition 6.2 have to be replaced as follows: (2) If the bottom stratum S_α is empty or if the bottom layer L_α *has no least two-valued model*, then the construction stops and $M_P = N_\alpha \cup \neg R_\alpha$ is the (partial) weakly perfect model of P. (3) In the remaining case L_α *has a least two-valued model*, and we define $M_\alpha = H \cup \neg R_\alpha$, where H is the partial model of L_α corresponding to its least two-valued model, and the construction continues.

The original definition is more general due to the fact that every definite program has a least two-valued model. However, while the least two-valued model of a definite program can be obtained as the least fixed point of the monotonic (and even Scott-continuous) operator T_P^+, we know of no similar result, or general operator, for obtaining the least two-valued model, if existent, of progams which are not definite. The original definition therefore seems to be rather awkward, and indeed, for the definition of weakly stratified programs [PP90], the more general version was dropped in favour of requiring definite layers. So Definition 6.2 is an adaptation taking the original notion of weakly stratified program into account, and appears to be more natural. In the following, the notion of *weakly perfect model* will refer to Definition 6.2.

To be pedantic, there is another difference, namely that we have made explicit the sets R_α of Definition 6.2, which were only implicitly treated in the original definition. The result is the same.

We show next that Definition 6.1 indeed captures the weakly perfect model. The proof basically follows our proof scheme, with some alterations, and the monotonic construction which defines the weakly perfect model serves in place of a monotonic operator. The technical details of the proof are very involved.

6.4 Theorem Let P be a normal logic program with weakly perfect model M_P. Then M_P is the greatest model among all models I, for which there exists an I-partial level mapping l for P such that P satisfies (WS) with respect to I and l.

We prepare the proof of Theorem 6.4 by introducing some notation, which will make the presentation much more transparent. As for the proof of Theorem 5.5, we will consider level mappings which map into pairs (β, n) of ordinals, where $n \leq \omega$.

Let P be a normal logic program with (partial) weakly perfect model M_P. Then define the M_P-partial level mapping l_P as follows: $l_P(A) = (\beta, n)$, where $A \in S_\beta \cup R_\beta$ and n is least with

$A \in T^+_{L_\beta}\!\uparrow(n+1)$, if such an n exists, and $n = \omega$ otherwise. We observe that if $l_P(A) = l_P(B)$ then there exists α with $A, B \in S_\alpha \cup R_\alpha$, and if $A \in S_\alpha \cup R_\alpha$ and $B \in S_\beta \cup R_\beta$ with $\alpha < \beta$, then $l(A) < l(B)$.

The following definition is again technical and will help to ease notation and arguments.

6.5 Definition Let P and Q be two programs and let I be an interpretation.

1. If $C_1 = (A \leftarrow L_1, \ldots, L_m)$ and $C_2 = (B \leftarrow K_1, \ldots, K_n)$ are two clauses, then we say that C_1 *subsumes* C_2, written $C_1 \preccurlyeq C_2$, if $A = B$ and $\{L_1, \ldots, L_m\} \subseteq \{K_1, \ldots, K_n\}$.

2. We say that P *subsumes* Q, written $P \preccurlyeq Q$, if for each clause C_1 in P there exists a clause C_2 in Q with $C_1 \preccurlyeq C_2$.

3. We say that P *subsumes* Q *model-consistently* (*with respect to* I), written $P \preccurlyeq_I Q$, if the following conditions hold. (i) For each clause $C_1 = (A \leftarrow L_1, \ldots, L_m)$ in P there exists a clause $C_2 = (B \leftarrow K_1, \ldots, K_n)$ in Q with $C_1 \preccurlyeq C_2$ and $(\{K_1, \ldots, K_n\} \setminus \{L_1, \ldots, L_m\}) \subseteq I$. (ii) For each clause $C_2 = (B \leftarrow K_1, \ldots, K_n)$ in Q with $\{K_1, \ldots, K_n\} \in I$ and $B \notin I$ there exists a clause C_1 in P such that $C_1 \preccurlyeq C_2$.

A clause C_1 subsumes a clause C_2 if both have the same head and the body of C_2 contains at least the body literals of C_1, e.g. $p \leftarrow q$ subsumes $p \leftarrow q, \neg r$. A program P subsumes a program Q if every clause in P can be generated this way from a clause in Q, e.g. the program consisting of the two clauses $p \leftarrow q$ and $p \leftarrow r$ subsumes the program consisting of $p \leftarrow q, \neg s$ and $p \leftarrow r, p$. This is also an example of a model-consistent subsumption with respect to the interpretation $\{\neg s, p\}$. Concerning Example 6.3, note that $P/N_1 \preccurlyeq_{N_1} P$, which is no coincidence. Indeed, Definition 6.5 facilitates the proof of Theorem 6.4 by employing the following lemma.

6.6 Lemma With notation from Definiton 6.2, we have $P/N_\alpha \preccurlyeq_{N_\alpha} P$ for all α.

Proof: Condition 3(i) of Definition 6.5 holds because every clause in P/N_α is obtained from a clause in P by deleting body literals which are contained in N_α. Condition 3(ii) holds because for each clause in P with head $A \notin N_\alpha$ whose body is true under N_α, we have that $A \leftarrow$ is a fact in P/N_α. ■

The next lemma establishes the induction step in part (2) of the proof of Theorem 6.4.

6.7 Lemma If I is a non-empty model of a (infinite propositional normal) logic program P' and l an I-partial level mapping such that P' satisfies (WS) with respect to I and l, then the following hold for $P = P'/\emptyset$.

(a) The bottom stratum $S(P)$ of P is non-empty and consists of trivial components only.

(b) The bottom layer $L(P)$ of P is definite.

(c) The definite (partial) model N of $L(P)$ is consistent with I in the following sense: we have $I' \subseteq N$, where I' is the restriction of I to all atoms which are not undefined in N.

(d) P/N satisfies (WS) with respect to $I \setminus N$ and l/N, where l/N is the restriction of l to the atoms in $I \setminus N$.

Proof: (a) Assume there exists some component $C \subseteq S(P)$ which is not trivial. Then there must exist atoms $A, B \in C$ with $A < B$, $B < A$, and $A \neq B$. Without loss of generality, we can assume that A is chosen such that $l(A)$ is minimal. Now let A' be any atom occuring in a clause with head A. Then $A > B > A \geq A'$, hence $A > A'$, and by minimality of the component we must also have $A' > A$, and we obtain that all atoms occuring in clauses with head A must be contained in C. We consider two cases.

(Case i) If $A \in I$, then there must be a fact $A \leftarrow$ in P, since otherwise by (WSi) we had a clause $A \leftarrow L_1, \dots, L_n$ (for some $n \geq 1$) with $L_1, \dots, L_n \in I$ and $l(A) > l(L_i)$ for all i, contradicting the minimality of $l(A)$. Since $P = P'/\emptyset$ we obtain that $A \leftarrow$ is the only clause in P with head A, contradicting the existence of $B \neq A$ with $B < A$.

(Case ii) If $\neg A \in I$, and since A was chosen minimal with respect to l, we obtain that condition (WSiib) must hold for each clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ with respect to I and l, and that $m = 0$. Furthermore, all A_i must be contained in C, as already noted above, and $l(A) \geq l(A_i)$ for all i by (WSiib). Also from (Case i) we obtain that no A_i can be contained in I. We have now established that for all A_i in the body of any clause with head A, we have $l(A) = l(A_i)$ and $\neg A_i \in I$. The same argument holds for all clauses with head A_i, for all i, and the argument repeats. Now from $A > B$ we obtain that there are $D, E \in C$ with $A \geq E$ (or $A = E$), $D \geq B$ (or $D = B$), and E refers negatively to D. As we have just seen, we obtain $\neg E \in I$ and $l(E) = l(A)$. Since E refers negatively to D, there is a clause with head E and $\neg D$ contained in the body of this clause. Since (WSii) holds for this clause, there must be a literal L in the body with level less than $l(E)$, hence $l(L) < l(A)$ and $L \in C$ which is a contradiction. We thus have established that all components are trivial.

We show next that the bottom stratum is non-empty. Indeed, let A be an atom such that $l(A)$ is minimal. We will show that $\{A\}$ is a component. So assume it is not, i.e. that there is B with $B < A$. Then there exist $D_1, \dots, D_k$, for some $k \in \mathbb{N}$, such that $D_1 = A$, D_j refers to D_{j+1} for all $j = 1, \dots, k-1$, and D_k refers negatively to some B' with $B' \geq B$ (or $B' = B$).

We show next by induction that for all $j = 1, \dots, k$ the following statements hold: $\neg D_j \in I$, $B < D_j$, and $l(D_j) = l(A)$. Indeed note that for $j = 1$, i.e. $D_j = A$, we have that $B < D_j = A$ and $l(D_j) = l(A)$. Assuming $A \in I$, we obtain by minimality of $l(A)$ that $A \leftarrow$ is the only clause in $P = P'/\emptyset$ with head A, contradicting the existence of $B < A$. So $\neg A \in I$, and the assertion holds for $j = 1$. Now assume the assertion holds some $j < k$. Then obviously $D_{j+1} > B$. By $\neg D_j \in I$ and $l(D_j) = l(A)$, we obtain that (WSii) must hold, and by the minimality of $l(A)$ we infer that (WSiib) must hold and that no clause with head D_j contains negated atoms. So $l(D_{j+1}) = l(D_j) = l(A)$ holds by (WSiib) and minimality of $l(A)$. Furthermore, the assumption $D_{j+1} \in I$ can be rejected by the same argument as for A above, because then $D_{j+1} \leftarrow$ would be the only clause with head D_{j+1}, by minimality of $l(D_{j+1}) = l(A)$, contradicting $B < D_{j+1}$. This concludes the inductive proof.

Summarizing, we obtain that D_k refers negatively to B', and that $\neg D_k \in I$. But then there is a clause with head D_k and $\neg B'$ in its body which satisfies (WSii), contradicting the minimality of $l(D_k) = l(A)$. This concludes the proof of statement (a).

(b) According to [PP90] we have that whenever all components are trivial, then the bottom layer is definite. So the assertion follows from (a).

(c) Let $A \in I'$ be an atom with $A \notin N$, and assume without loss of generality that A is chosen such that $l(A)$ is minimal with these properties. Then there must be a clause $A \leftarrow \texttt{body}$ in P such that all literals in $\texttt{body}$ are true with respect to I', hence with respect to N by minimality of $l(A)$. Thus $\texttt{body}$ is true in N, and since N is a model of $L(P)$ we obtain $A \in N$, which contradicts our assumption.

Now let $A \in N$ be an atom with $A \notin I'$, and assume without loss of generality that A is chosen such that n is minimal with $A \in T^+_{L(P)} \uparrow (n+1)$. But then there is a definite clause $A \leftarrow \texttt{body}$ in $L(P)$ such that all atoms in $\texttt{body}$ are true with respect to $T^+_{L(P)} \uparrow n$, hence also with respect to I', and since I' is a model of $L(P)$ we obtain $A \in I'$, which contradicts our assumption.

Finally, let $\neg A \in I'$. Then we cannot have $A \in N$ since this implies $A \in I'$. So $\neg A \in N$ since N is a total model of $L(P)$.

(d) From Lemma 6.6, we know that $P/N \preccurlyeq_N P$. We distinguish two cases.

(Case i) If $I \setminus N \models A$, then there must exist a clause $A \leftarrow L_1, \dots, L_k$ in P such that $L_i \in I$ and $l(A) > l(L_i)$ for all i. Since it is not possible that $A \in N$, there must also be a clause in P/N which subsumes $A \leftarrow L_1, \dots, L_k$, and which therefore satisfies (WSi). So A satisfies (WSi).

(Case ii) If $\neg A \in I \setminus N$, then for each clause $A \leftarrow \texttt{body1}$ in P/N there must be a clause $A \leftarrow \texttt{body}$ in P which is subsumed by the former, and since $\neg A \in I$, we obtain that condition (WSii) must be satisfied by A, and by the clause $A \leftarrow \texttt{body}$. Since reduction with respect to N removes only body literals which are true in N, condition (WSii) is still met. ■

We can now proceed with the proof.

Proof of Theorem 6.4: The proof will be established by showing the following facts: (1) P satisfies (WS) with respect to M_P and l_P. (2) If I is a model of P and l an I-partial level mapping such that P satisfies (WS) with respect to I and l, then $I \subseteq M_P$.

(1) Let $A \in \mathsf{dom}(l_P)$ and $l_P(A) = (\alpha, n)$. We consider two cases.

(Case i) If $A \in M_P$, then $A \in T^+_{L_\alpha} \uparrow (n+1)$. Hence there exists a definite clause $A \leftarrow A_1, \dots, A_k$ in L_α with $A_1, \dots, A_k \in T^+_{L_\alpha} \uparrow n$, so $A_1, \dots, A_k \in M_P$ with $l_P(A) > l_P(A_i)$ for all i. Since $P/N_\alpha \preccurlyeq_{N_\alpha} P$ by Lemma 6.6, there must exist a clause $A \leftarrow A_1, \dots, A_k, L_1, \dots, L_m$ in P with literals $L_1, \dots, L_m \in N_\alpha \subseteq M_P$, and we obtain $l_P(L_j) < l_P(A)$ for all $j = 1, \dots, m$. So (WSi) holds in this case.

(Case ii) If $\neg A \in M_P$, then let $A \leftarrow A_1, \dots, A_k, \neg B_1, \dots, \neg B_m$ be a clause in P, noting that (WSii) is trivially satisfied in case no such clause exists. We consider the following two subcases.

(Subcase ii.a) Assume A is undefined in N_α and was eliminated from P by reducing it with respect to N_α, i.e. $A \in R_\alpha$. Then, in particular, there must be some $\neg A_i \in N_\alpha$ or some $B_j \in N_\alpha$, which yields $l_P(A_i) < l_P(A)$, respectively $l_P(B_j) < l_P(A)$, and hence one of (WSiia), (WSiic) holds.

(Subcase ii.b) Assume $\neg A \in H$, where H is the definite (partial) model of L_α. Since P/N_α subsumes P model-consistently with respect to N_α, we obtain that there must be some A_i with $\neg A_i \in H$, and by definition of l_P we obtain $l_P(A) = l_P(A_i) = (\alpha, \omega)$, and hence also $l_P(A_{i'}) \leq l_P(A_i)$ for all $i' \neq i$. Furthermore, since P/N_α is definite, we obtain that $\neg B_j \in N_\alpha$ for all j, hence $l_P(B_j) < l_P(A)$ for all j. So condition (WSiib) is satisfied.

(2) First note that for all models M, N of P with $M \subseteq N$ we have $(P/M)/N = P/(M \cup N) = P/N$ and $(P/N)/\emptyset = P/N$.

Let I_α denote I restricted to the atoms which are not undefined in $N_\alpha \cup R_\alpha$. It suffices to show the following: For all $\alpha > 0$ we have $I_\alpha \subseteq N_\alpha \cup R_\alpha$, and $I \setminus M_P = \emptyset$.

We next show by induction that if $\alpha > 0$ is an ordinal, then the following statements hold. (a) The bottom stratum of P/N_α is non-empty and consists of trivial components only. (b) The bottom layer of P/N_α is definite. (c) $I_\alpha \subseteq N_\alpha \cup R_\alpha$. (d) $P/N_{\alpha+1}$ satisfies (WS) with respect to $I \setminus N_{\alpha+1}$ and $l/N_{\alpha+1}$.

Note first that P satisfies the hypothesis of Lemma 6.7, hence also its consequences. So $P/N_1 = P/\emptyset$ satisfies (WS) with respect to $I \setminus N_1$ and l/N_1, and by application of Lemma 6.7 we obtain that statements (a) and (b) hold. For (c), note that no atom in R_1 can be true in I, because no atom in R_1 can appear as head of a clause in P, and apply Lemma 6.7 (c). For (d), apply Lemma 6.7, noting that $P/N_2 \preccurlyeq_{N_2} P$.

For α being a limit ordinal, we can show exactly as in the proof of Lemma 6.7 (d), that P satisfies (WS) with respect to $I \setminus N_\alpha$ and l/N_α. So Lemma 6.7 is applicable and statements (a) and (b) follow. For (c), let $A \in R_\alpha$. Then every clause in P with head A contains a body literal which is false in N_α. By induction hypothesis, this implies that no clause with head A in P can have a body which is true in I. So $A \notin I$. Together with Lemma 6.7 (c), this proves statement (c). For (d), apply again Lemma 6.7 (d), noting that $P/N_{\alpha+1} \preccurlyeq_{N_{\alpha+1}} P$.

For $\alpha = \beta + 1$ being a successor ordinal, we obtain by induction hypothesis that P/N_β satisfies the hypothesis of Lemma 6.7, so again statements (a) and (b) follow immediately from this lemma, and (c), (d) follow as in the case for α being a limit ordinal.

It remains to show that $I \setminus M_P = \emptyset$. Indeed by the transfinite induction argument just given we obtain that P/M_P satisfies (WS) with respect to $I \setminus M_P$ and l/M_P. If $I \setminus M_P$ is non-empty, then by Lemma 6.7 the bottom stratum $S(P/M_P)$ is non-empty and the bottom layer $L(P/M_P)$ is definite with definite (partial) model M. Hence by definition of the weakly perfect model M_P of P we must have that $M \subseteq M_P$ which contradicts the fact that M is the definite model of $L(P/M_P)$. Hence $I \setminus M_P$ must be empty which concludes the proof. ■

Of independent interest is again the case, where the model in question is total. We see immediately, for example, in the light of Theorem 3.3, that the model is then stable.

6.8 Corollary A normal logic program P is weakly stratified, i.e. has a total weakly perfect model, if and only if there is a total model I of P and a (total) level mapping l for P such that P satisfies (WS) with respect to I and l.

We also obtain the following corollary as a trivial consequence of our uniform characterizations by level mappings.

6.9 Corollary Let P be a normal logic progam with Fitting model M_{F}, weakly perfect model M_{WP}, and well-founded model M_{WF}. Then $M_{\mathrm{F}} \subseteq M_{\mathrm{WP}} \subseteq M_{\mathrm{WF}}$.

6.10 Example Consider the program P from Example 6.10. Then $M_{\mathrm{F}} = \emptyset$, $M_{\mathrm{WP}} = \{\neg d, \neg e\}$, and $M_{\mathrm{WF}} = \{a, \neg b, \neg c, \neg d, \neg e\}$.

7 Related Work

As already mentioned in the introduction, level mappings have been used for studying semantic aspects of logic programs in a number of different ways. Our presentation suggests a novel application of level mappings, namely for providing uniform characterizations of different fixed-point semantics for logic programs with negation. Although we believe our perspective to be new in this general form, there nevertheless have been results in the literature which are very close in spirit to our characterizations.

A first noteable example of this is Fages' characterization of stable models [Fag94], which we have stated in Theorem 3.3. Another result which uses level mappings to characterize a semantics is by Lifschitz, Przymusinski, Stärk, and McCain [LMPS95, Lemma 3]. We briefly compare their characterization of the well-founded semantics and ours. In fact, this discussion can be based upon two different characterizations of the least fixed point of a monotonic operator F. On the one hand, this least fixed point is of course the *least of all fixed points* of F, and on the other hand, this least fixed point is the limit of the sequence of powers $(F \uparrow \alpha)_\alpha$, and in this latter sense is the *least iterate* of F *which is also a fixed point*. Our characterizations of definite, Fitting, well-founded, and weakly stratified semantics use the latter approach, which is reflected in our general proof scheme, which defines level mappings according to powers, or iterates, of the respective operators. The results by Fages [Fit94] and Lifschitz et al. [LMPS95] hinge upon the former approach, i.e. they are based on the idea of characterizing the fixed points of an operator — GL_P, respectively Ψ_P [Prz89, BNN91] — and so the sought fixed point turns out to be the least of those. Consequently, as can be seen in the proof of Theorem 3.3, the level mapping in Fages' characterization, and likewise in the result by Lifschitz et al., arises only indirectly from the operator — GL_P, respectively Ψ_P — whose fixed point is sought. Indeed, the level mapping by Fages is defined according to iterates of $T_{P/I}$, which is the operator for obtaining $\mathrm{GL}_P(I)$, for any I. The result by Lifschitz et al. is obtained similarly based on a three-valued operator Ψ_P.

Unforunately, these characterizations by Fages, in Theorem 3.3, respectively by Lifschitz et al. [LMPS95], seem to be applicable only to operators which are defined by least fixed points of other operators, as is the case for GL_P and Ψ_P, and it seems that the approach by Lifschitz et al. is unlikely to scale to other semantics. For example, we attempted a straightforward characterization of the Fitting semantics in the spririt of Lifschitz et al. which failed.

On a more technical level, a difference between our result, Theorem 5.2, and the characterization by Lifschitz et al. [LMPS95] of the well-founded semantics is this: In our characterization, the model is described using conditions on atoms which are true or false (i.e. *not undefined*) in the well-founded model, whereas in theirs the conditions are on those atoms which are true or undefined (i.e. *not false*) in the well-founded model. The reason for this is that we consider iterates of W_P, where $W_P \uparrow 0 = \emptyset$, while they use the fact that each fixed point of Ψ_P is a least fixed point of $\Phi_{P/I}$ with respect to the truth ordering on interpretations (note that in this case P/I denotes a three-valued generalization of the Gelfond-Lifschitz transformation due to Przymusinski [Prz89]). In this ordering we have $\Phi_{P/I} \uparrow 0 = \neg B_P$. It is nevertheless nice to note that in the special case of the well-founded semantics there exist two complementary characterizations using level mappings.

Since our proposal emphasizes uniformity of characterizations, it is related to the large body of work on uniform approaches to logic programming semantics, of which we will discuss two in more detail: the algebraic approach via bilattices due to Fitting, and the work of Dix.

Bilattice-based semantics has a long tradition in logic programming theory, starting out from the four-valued logic of Belnap [Bel77]. The underlying set of truth values, a four-element lattice, was recognized to admit two ordering relations which can be interpreted as truth- and knowledge-order. As such it has the structure of a bilattice, a term due to Ginsberg [Gin86], who was the first to note the importance of bilattices for inference in artificial intelligence [Gin92]. This general approach was imported into logic programming theory by Fitting [Fit91a]. Although multi-valued logics had been used for logic programming semantics before [Fit85], bilattices provided an interesting approach to semantics as they are capable of incorporating both reasoning about truth and reasoning about knowledge, and, more technically, because they have nice algebraic behaviour. Using this general framework Fitting was able to show interesting relationships between the stable and the well-founded semantics [Fit91b, Fit93, Fit02].

Without claiming completeness we note two current developments in the bilattice-based approach to logic programming: Fitting's framework has been extended to an algebraic approach for approximating operators by Denecker, Marek, and Truszczynski [DMT00]. The inspiring starting point of this work was the noted relationship between the stable model semantics and the well-founded semantics, the latter approximating the former. The other line of research was pursued mainly by Avron and Arieli [AA94, AA98, Ari02], who use bilattices for paraconsistent reasoning in logic programming. The above outline of the historical development of bilattices in logic programming theory suggests a similar kind of uniformity as we claim for our approach. The exact relationship between both approaches, however, is still to be investigated. On the one hand, bilattices can cope with paraconsistency — an issue of logic programming and deductive databases, which is becoming more and more important — in a very convenient way. On the other hand, our approach can deal with semantics based on multi-valued logics, whose underlying truth structure is not a bilattice. A starting point for investigations in this direction could be the obvious meeting point of both theories: the well-founded semantics for which we can provide a characterization and which is a special case of the general approximation theory of Denecker et al. [DMT00].

Another very general, and uniform, approach to logic programming pursues a different point of view, namely logic programming semantics as nonmonotonic inference. The general theory of nonmonotonic inference and a classification of properties of nonmonotonic operators was developed by Kraus, Lehmann, and Magidor [KLM90], leading to the notion *KLM-axioms* for these properties, and developed further by Makinson [Mak94]. These axioms were adopted to the terminology of logic programming and extended to a general theory of logic programming semantics by Dix [Dix95a, Dix95b]. In this framework, different known semantics are classified according to strong properties — the KLM-axioms which hold for the semantics – and weak properties — specific properties which deal with the irregularities of negation-as-failure. As such Dix' framework is indeed a general and uniform approach to logic programming, its main focus being on semantic properties of logic programs. Our approach in turn could be called *semi-syntactic* in that definitions that employ level mappings naturally take the structure of the logic program into account. As in the case of the bilattice-based approaches, it is not yet completely clear whether these two approaches can be amalgamated in the sense of a correspondence between properties of level mappings, e.g. strict or semi-strict descent of the level, etc., on the one hand, and KLM-properties of the logic program on the other. However, we believe that it is possible to develop a proof scheme for nonmonotonic properties of logic programs in the style of the proof scheme presented in the paper, which can be used to cast semantics based on monotonic operators into level mapping form.

We finally mention the work by Hitzler and Seda [HS99], which was the root and starting point for our investigations. This framework aims at the characterization of program classes, such as (locally) stratified programs [ABW88, Prz88], acceptable programs [AP93], or Φ-accessible programs [HS99]. Such program classes appear naturally whenever a semantics is not defined for all logic programs. In these cases one tries to characterize those programs, for which the semantics is well-defined or well-behaved. Their main tool were monotonic operators in three-valued logic, in the spirit of Fitting's Φ_P, rather than level mappings. With each operator comes a least fixed point, hence a semantics, and it is easily checked that these semantics can be characterized using our approach, again by straightforward application of our proof scheme. Indeed, preliminary steps in this direction already led to an independent proof of a special case of Corollary 6.9 [HS01].

8 Conclusions and Further Work

We have proposed a novel approach for obtaining uniform characterizations of different semantics for logic programs. We have exemplified this by giving new alternative characterizations of some of the major semantics from the literature. We have developed and presented a methodology for obtaining characterizations from monotonic semantic operators or related constructions, and a proof scheme for showing correctness of the obtained characterizations. We consider our contribution to be fundamental, with potential for extension in many directions.

Our approach employs level mappings as central tool. The uniformity with which our characterizations were obtained and proven to be correct suggests that our method should be of wider applicability. In fact, since it builds upon the well-known Tarski fixed point theorem, it should scale well to most, if not all semantics, which are defined by means of a monotonic operator. The main contribution of this paper is thus, that we have developed a novel way of presenting logic programming semantics in some kind of *normal* or *standard form*. This can be used for easy comparison of semantics with respect to the syntactic structures that can be used with a certain semantics, i.e. to what extent the semantics is able to 'break up' positive or negative dependencies or loops between atoms in the program, as in Corollary 6.9.

However, there are many more requirements which a general and uniform approach to logic program semantics should eventually be able to meet, including (i) a better understanding of known semantics, (ii) proof schemes for deriving properties of semantics, (iii) extendability to new programming constructs, and (iv) support for designing new semantics for special purposes.

Requirement (i) is met to some extent by our appoach, since it enables easy comparison of semantics, as discussed earlier. However, in order to meet the other requirements, i.e. to set up a *meta-theory* of level-mapping-based semantics, a lot of further research is needed. We list some topics to be pursued in the future, some of which are under current investigation by the authors. There are many properties which are interesting to know about a certain semantics, depending on one's perspective. For the nonmonotonic reasoning aspect of logic programming it would certainly be interesting to have a proof scheme as flexible and uniform as the one presented in this paper. Results and proofs in the literature [Fag94, Dix95a, Tur01] suggest that there is a strong dependency between notions of ordering on the Herbrand base, as expressed by level mappings, and KLM-properties satisfied by a semantics, which constitutes some evidence that a general proof scheme for proving KLM-properties from level mapping definitions can be developed. Other interesting properties are e.g. the computational complexity of a semantics, but also logical characterizations of the behaviour of negation in logic programs, a line of research initiated by Pearce [Pea97].

For (iii), it would be desirable to extend our characterizations also to disjunctive programs, which could perhaps contribute to the discussion about appropriate generalizations of semantics of normal logic programs to the disjunctive case.

We finally want to mention that the elegant mathematical framework of level mapping definitions naturally gives rise to the design of new semantics. However, at the time being this is only a partial fulfillment of (iv): As long as a meta-theory for level-mapping-based semantics is missing, one still has to apply conventional methods for extracting properties of the respective semantics from its definition.

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