Dislocated Topologies*

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Abstract

We study a generalized notion of topology which evolved out of applications in the area of logic programming semantics. The generalization is obtained by relaxing the requirement that a neighbourhood of a point includes the point itself, and by allowing neighbourhoods of points to be empty. The corresponding generalized notion of metric is obtained by allowing points to have non-zero distance to themselves. We further show that it is meaningful to discuss neighbourhoods, convergence, and continuity in these spaces. A generalized version of the Banach contraction mapping theorem can also be established. We show finally how the generalized metrics studied here can be obtained from conventional metrics.

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1 Introduction

In recent years, the rôle of topology in Logic Programming has come to be recognized (see e.g. [BS89, BDJRS99, Fit94, HS99, HS00, HSK99, KKM93, PR00, Sed95, Sed97, SH97, SH98]). In particular, topological methods are employed in order to obtain fixed-point semantics for logic programs. The *dislocated metric spaces* which we discuss in this paper are motivated by such considerations. Whilst the main part of the paper consists of an analysis of these spaces, we find it convenient to first bring to the attention of the reader the general perspective from which they emerge. The reader who is not interested in the motivational background from the theory of logic programming may go directly to Section 2.1 and then on to Section 3.

In the classical approach to logic programmming semantics in which definite or positive programs P are considered (those in which negation does not occur), we associate an operator T_P, called the *single-step* or *immediate consequence operator*, see [Llo88] and also Section 2.2. This operator turns out to be continuous with respect to the Scott topology on the complete lattice of all interpretations. Applying the Knaster-Tarski fixed-point theorem[1] then yields a least fixed point of T_P, which is often understood to be the denotational semantics or meaning of the program in question. As it turns out, this semantics also agrees very well with the operational and the logical readings of the program, see [Llo88] again.

However, when the syntax is enhanced in the sense that negation is allowed, resulting in the class of *normal logic programs*, then the single-step operator is no longer monotonic and therefore cannot be continuous in the Scott topology[2]. Hence, the approach mentioned above using the Knaster-Tarski theorem is invalid and other methods have to be sought. These include (1) restricting the syntax of the programs in question (e.g. in [ABW88, Cav89, Prz88, SH97]), (2) using alternative operators (e.g. in [Fit85, GRS91, GL88, HS99]), and (3) applying alternative fixed-point theorems in order to analyse non-monotonic operators. It is this latter point (3) which provides the motivation for the results presented in this paper.

The main alternative to the Knaster-Tarski theorem is the Banach contraction mapping theorem for complete metric spaces. In some cases, e.g. for acyclic[3] programs, the

[1] See [SLG94] for domain theoretic background.

[2] In fact, the appropriate topology for normal logic programs appears to be the *atomic topology* of [Sed95], which is a Cantor topology and a generalization of the *query topology* presented in [BS89].

[3] Called ω*-locally hierarchical in [Cav89].*

Banach theorem can indeed be applied, cf. [SH98, HSK99]. Acyclic programs, however, are a rather restrictive class and, furthermore, the topological spaces which arise in the area of denotational semantics are often not Hausdorff. It is therefore of interest to find fixed-point theorems for spaces which are weaker than metric spaces in a topological sense. The alternatives include (a) quasi-metrics, where the symmetry axiom is dropped (cf. [Sed97]), which in fact have recently been studied extensively in domain theory, and (b) generalized metric spaces, generalized in the sense that the target space of the distance function is a partially ordered set, cf. [KKM93, SH97, PR00]. We wish to put forward a third alternative (c) which we call *dislocated metrics*, and which are obtained by omitting the requirement that the distance from a point to itself must be zero. This idea is not new and has been studied in the context of domain theory in [Mat86, Mat92a, Mat92b], where the latter two references focus on slightly stronger spaces, cf. Section 3, and dislocated metrics were called *metric domains* in the first. Our motivation for studying these spaces is that a generalization of the Banach contraction mapping theorem can indeed be established [Mat86] and applications to logic programming semantics can be found (see Section 2.2).

The plan of the paper is as follows. In Section 2, we will first define *dislocated metrics* and state the generalization of the Banach contraction mapping theorem we require; then we will review one of the applications of these notions to logic programming semantics. In Section 3, we will study *dislocated topologies* which generalize conventional topologies and can be thought of as underlying the notion of dislocated metric. In particular, we will see that it is meaningful to talk about neighbourhoods, convergence, and continuity in these spaces. In Section 4, we will further investigate dislocated metrics and their relationships with the notions from Section 3 and with conventional metrics. Finally, in Section 5, we will conclude with a short discussion.

2 Motivation: A Fixed Point Application in Logic Programming

We will define dislocated metrics, and present a generalization of the Banach contraction mapping theorem, Theorem 2.7, for these spaces. Then we will discuss acceptable logic programs and apply Theorem 2.7 in order to obtain a unique supported model for these programs.

2.1 A Generalized Banach Contraction Mapping Theorem

2.1 Definition Let X be a set and let $\varrho : X \times X \to \mathbb{R}_0^+$ be a function, called a *distance function*. Consider the following conditions:

(Mi) For all $x \in X$, $\varrho(x,x) = 0$.

(Mii) For all $x, y \in X$, if $\varrho(x,y) = 0$ then $x = y$.

(Miii) For all $x, y \in X$, $\varrho(x, y) = \varrho(y, x)$.

(Miv) For all $x, y, z \in X$, $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$.

(Miv$'$) For all $x, y, z \in X$, $\varrho(x, y) \leq \max\{\varrho(x, z), \varrho(z, y)\}$.

If ϱ satisfies conditions (Mi) to (Miv), then it is called a *metric*. If it satisfies conditions (Mi), (Miii) and (Miv), it is called a *pseudo-metric*. If it satisfies (Mii), (Miii) and (Miv), we will call it a *dislocated metric* (or simply *d-metric*). This terminology will become clearer later on. If a (pseudo-, d-) metric satisfies the strong triangle inequality (Miv$'$), then it is called a (pseudo-, d-) *ultra*metric.

As already mentioned, dislocated metrics were studied under the notion of *metric domains* in [Mat86]. We will take this up again in Section 3, but proceed now with the definitions needed for stating the generalized Banach contraction mapping theorem, that is, we will define convergence, Cauchy sequences and completeness for dislocated metrics. As it turns out, these notions can be carried over directly from conventional metrics.

2.2 Definition A sequence (x_n) in a d-metric space (X, ϱ) *converges with respect to* ϱ (or *in* ϱ) if there exists an $x \in X$ such that $\varrho(x_n, x)$ converges to 0 as $n \to \infty$. In this case, x is called the *limit* of (x_n) (*in* ϱ) and we write $x_n \to x$.

2.3 Proposition Limits in d-metric spaces are unique.

Proof: Let x and y be limits of the sequence (x_n). By properties (Mii) and (Miii) of Definition 2.1, it follows that $\varrho(x, y) \leq \varrho(x_n, x) + \varrho(x_n, y) \to 0$ as $n \to \infty$. Hence $\varrho(x, y) = 0$ and by property (Mii) of Definition 2.1 it follows that $x = y$. ■

2.4 Definition A sequence (x_n) in a d-metric space is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $\varrho(x_m, x_n) < \varepsilon$.

2.5 Proposition Every converging sequence in a d-metric space is a Cauchy sequence.

Proof: Let (x_n) be a sequence which converges to some x, and let $\varepsilon > 0$ be arbitrarily chosen. Then there exists $n_0 \in \mathbb{N}$ with $\varrho(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. For $m, n \geq n_0$ we then obtain $\varrho(x_m, x_n) \leq \varrho(x_m, x) + \varrho(x, x_n) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. Hence (x_n) is a Cauchy sequence. ■

2.6 Definition A d-metric space (X, ϱ) is called *complete* if every Cauchy sequence in X converges with respect to ϱ. A function $f : X \to X$ is called a *contraction* if there exists $0 \leq \lambda < 1$ such that $\varrho(f(x), f(y)) \leq \lambda\varrho(x, y)$ for all $x, y \in X$.

2.7 Theorem Let (X, ϱ) be a complete d-metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point.

A proof of this theorem was given in [Mat86]. We will give an alternative proof in Section 4 which is more in the spirit of the proof of the original Banach contraction mapping theorem, and proceed now with an application of it.

2.2 Acceptable Logic Programs

We sketch an application of Theorem 2.7 in the area of logic programming semantics. Our main reference for logic programming is [Llo88].

A *logic program* is a finite set of (universally quantified) clauses, from first order logic, of the form

$$\forall\,(A \leftarrow L_1 \wedge \cdots \wedge L_n)$$

where A is an atom and $L_1, \ldots, L_n$ are literals. Such clauses are usually written as

$$A \leftarrow L_1, \ldots, L_n.$$

We call A the *head* of the clause, and $L_1 \ldots, L_n$ the *body* of the clause. Each L_i is called a *body literal* of the clause. If $n = 0$, then the clause is called a *unit clause* or a *fact*.

For a given logic program P, we denote the Herbrand base (i.e. the set of all ground atoms in the underlying first order language) by B_P. As usual, (Herbrand-) interpretations of P can be identified with subsets of B_P, so that the set I_P of all interpretations of P coincides with the power set $\mathcal{P}(B_P)$ of B_P.

The standard approach to logic programming semantics, that is, to assigning a reasonable meaning to a given logic program, is to identify models of the program which have certain additional properties. We will focus on the *supported model semantics* or *Clark completion semantics* of P, cf. [Cla78, ABW88]. For this purpose, we define the *immediate consequence* or *single step operator* T_P for a given logic program P as a mapping $T_P : I_P \rightarrow I_P$ of interpretations to interpretations as follows: $T_P(I)$ is the set of all $A \in B_P$ such that there exists a ground instance $A \leftarrow L_1, \ldots, L_n$ of a clause in P, with head A, satisfying $I \models L_1 \wedge \cdots \wedge L_n$. Note that T_P is in general not monotonic but is so if P is definite.

As it turns out, the models of P are exactly the pre-fixed points of T_P, that is, those interpretations which satisfy $T_P(I) \subseteq I$. A *supported model* (or model of the Clark completion [Cla78]) of P is a fixed point of T_P.

The following definition is taken from [AP93]. Acceptable programs play an important rôle in termination analysis in logic programming.

2.8 Definition Let P be a logic program and let p, q be predicate symbols occurring in P.

1. p *refers to* q if there is a clause in P with p in its head and q in its body.

2. p *depends on* q if (p, q) is in the reflexive, transitive closure of the relation *refers to*.

3. Neg_P denotes the set of predicate symbols in P which occur in a negative literal in the body of a clause in P.

4. Neg_P^* denotes the set of all predicate symbols in P on which the predicate symbols in Neg_P depend.

5. P^- denotes the set of clauses in P whose head contains a predicate symbol from Neg_P^*.

Let P be a logic program, let $l : B_P \to \mathbb{N}$ be a *level mapping* and let I be a model of P whose restriction to the predicate symbols in Neg_P^* is a supported model of P^-. Then P is called *acceptable* (with respect to l and I) provided that the following condition holds. For each ground instance $A \leftarrow L_1, \ldots, L_n$ of a clause in P and for all $i \in \{1, \ldots, n\}$ we have:

$$\text{if} \qquad I \models \bigwedge_{j=1}^{i-1} L_j, \qquad \text{then} \qquad l(A) > l(L_i).$$

In the following, P is an acceptable program which satisfies the defining conditions with respect to a model I and a level mapping l.

For $J, K \in I_P$, we now define $d(K, K) = 0$ and $d(J, K) = 2^{-n}$, where J and K differ on some atom $A \in B_P$ of level n, but agree on all ground atoms of lower level. As was shown in [Fit94], (I_P, d) is a complete metric space, in fact even an ultrametric space. Next, we define a function $f : I_P \to \mathbb{R}$ by $f(K) = 0$ if $K \subseteq I$ and, if $K \not\subseteq I$, $f(K) = 2^{-n}$, where n is the smallest integer such that there is an atom $A \in B_P$ with $l(A) = n$, $K \models A$ and $I \not\models A$. Finally, we define $u : I_P \to \mathbb{R}$ by $u(K) = \max\{f(K'), d(K', I)\}$, where K' is K restricted to the predicate symbols which are not in Neg_P^*, and we define $\varrho : I_P \times I_P \to \mathbb{R}$ by

$$\varrho(J, K) = \max\{d(J, K), u(J), u(K)\}.$$

As it turns out, see Proposition 4.11 and Lemma 4.12, ϱ is a complete d-ultrametric on I_P, but not a metric (contrary to [Fit94], see also [HS00] for a discussion of this). Our definition of ϱ also differs slightly from [Fit94], but pointwise equality can easily be shown.

The following proposition was shown in [Fit94].

2.9 Proposition Let P be acceptable and let ϱ be defined as above. Then the associated immediate consequence operator T_P is a contraction on (I_P, ϱ).

By Theorem 2.7 we can therefore conclude the following theorem.

2.10 Theorem Each acceptable program has a unique supported model.

We will further discuss this result in Section 4.

3 Dislocated Topologies

We are interested in investigating the topological point of view of dislocated metrics following the outline of Section 2.1. Since constant sequences do not in general converge in d-metric spaces, a conventional topological approach is not feasable, and notions of neighbourhoods, convergence and continuity will have to be modified.

3.1 Neighbourhoods

3.1 Definition An (*open* ε-)*ball* in a d-metric space (X, ϱ) with *centre* $x \in X$ is a set $B_\varepsilon(x) = \{y \in X \mid \varrho(x, y) < \varepsilon\}$ where $\varepsilon > 0$.

Note that balls may be empty in d-metric spaces. In fact, the above definition of ball does not imply that the centre of a ball is contained in the ball itself: the point may be *dislocated* from the ball, and hence our usage of the term "dislocated" in this paper.

3.2 Proposition Let (X, ϱ) be a d-metric space.
(a) The following three conditions are equivalent:
(i) For all $x \in X$, we have $\varrho(x, x) = 0$.
(ii) ϱ is a metric.
(iii) For all $x \in X$ and all $\varepsilon > 0$, we have $B_\varepsilon(x) \neq \emptyset$.
(b) The space (X', ϱ), where $X' = \{x \in X \mid \varrho(x, x) = 0\}$, is a metric space.

Proof: (a) That (i) implies (ii) is obvious, as is (ii) implies (iii). We show (iii) implies (i). Since $B_\varepsilon(x) \neq \emptyset$ for all $\varepsilon > 0$, there exists, for each $\varepsilon > 0$, some $y \in X$ with $\varrho(x, y) < \varepsilon$. But, for all $y \in X$, we have $\varrho(x, x) \leq 2 \cdot \varrho(x, y)$, and hence $\varrho(x, x) < \varepsilon$ for all $\varepsilon > 0$. Therefore, $\varrho(x, x) = 0$.
(b) Obviously, (X', ϱ) is a d-metric space. The assertion now follows immediately from (a). ∎

We proceed next with the investigation of dislocated metrics from the topological point of view.

3.3 Definition Let X be a set. A relation $\lessdot \subseteq X \times \mathcal{P}(X)$ (written infix) is called a *d-membership relation (on X)* if it satisfies the following property for all $x \in X$ and $A, B \subseteq X$:

$$x \lessdot A \text{ and } A \subseteq B \text{ implies } x \lessdot B. \tag{1}$$

We say "x is below A" if $x \lessdot A$.

The "below"-relation is a generalization of the membership relation from set-theory, which will allow us to define a suitable notion of neighbourhood.

3.4 Definition Let X be a set, let $\lessdot$ be a d-membership relation on X and let $\mathcal{U}_x \neq \emptyset$ be a collection of subsets of X for each $x \in X$. We call $(\mathcal{U}_x, \lessdot)$ a *d-neighbourhood system (d-nbhood system) for x* if it satisfies the following conditions.

(Ni) If $U \in \mathcal{U}_x$, then $x \lessdot U$.

(Nii) If $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.

(Niii) If $U \in \mathcal{U}_x$, then there is a $V \subseteq U$ with $V \in \mathcal{U}_x$ such that for all $y \lessdot V$ we have $U \in \mathcal{U}_y$.

(Niv) If $U \in \mathcal{U}_x$ and $U \subseteq V$, then $V \in \mathcal{U}_x$.

Each $U \in \mathcal{U}_x$ is called a *d-neighbourhood* (*d-nbhood*) of x. Finally, let X be a set, let $\lessdot$ be a d-membership relation on X and, for each $x \in X$, let $(\mathcal{U}_x, \lessdot)$ be a d-nbhood system for x. Then $(X, \mathcal{U}, \lessdot)$ (or simply X) is called a *d-topological space*, where $\mathcal{U} = \{\mathcal{U}_x \mid x \in X\}$.

Note that points may have empty d-nbhoods and that Definition 3.4 is exactly the definition of a topological neighbourhood system if $\lessdot$ is the membership relation $\in$.

Proposition 3.5, next, shows that d-nbhood systems arise naturally from d-metrics.

3.5 Proposition Let (X, ϱ) be a d-metric space. Define the d-membership relation $\lessdot$ as the relation $\{(x, A) \mid \text{ there exists } \varepsilon > 0 \text{ for which } B_\varepsilon(x) \subseteq A\}$. For each $x \in X$, let $\mathcal{U}_x$ be the collection of all subsets A of X such that $x \lessdot A$. Then $(\mathcal{U}_x, \lessdot)$ is a d-nbhood system for x for each $x \in X$.

Proof: It is easy to see that $\lessdot$ is indeed a d-membership relation.
(Ni) is obvious. Note that we also have the reverse property: if $x \lessdot U$, then $U \in \mathcal{U}_x$.
(Nii) If $x \lessdot U, V$, then there are balls A, B with centre x such that $A \subseteq U$ and $B \subseteq V$. Without loss of generality let A be the smaller of the balls A and B. Then $A = A \cap B \subseteq U \cap V$.
(Niii) Let $U \in \mathcal{U}_x$, that is, $x \lessdot U$. Then there is a ball B with centre x such that $B \subseteq U$ and $B \in \mathcal{U}_x$. Now let $y \lessdot B$ be arbitrary. We have to show that $y \lessdot U$. But $y \lessdot B$ implies that there is a ball B' with centre y such that $y \lessdot B' \subseteq B \subseteq U$. So $y \lessdot U$.
(Niv) This is obvious since $x \lessdot U \subseteq V$ implies $x \lessdot V$. ■

We note that if (X, ϱ) is a metric space, then the above construction yields the usual topology associated with a metric.

The set of balls of a d-metric does not in general yield a conventional topology. In this respect, the axioms defining a dislocated metric are different from those defining a partial metric in [Mat92a, Mat92b], which are as follows.

3.6 Definition Let X be a set and let $p : X \times X \to \mathbb{R}_0^+$ be a function. We call p a *partial metric on* X if it satisfies the following axioms.

(Pi) For all $x, y \in X$, $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$.

(Pii) For all $x, y \in X$, $p(x, x) \leq p(x, y)$.

(Piii) For all $x, y \in X$, $p(x, y) = p(y, x)$.

(Piv) For all $x, y, z \in X$, $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

It is clear that any partial metric is a d-metric. Furthermore, the set of balls with respect to a partial metric does indeed yield a topology, and strong relationships between the topologies arising from partial metrics and topologies discussed in domain theory can be established. We refer the reader to [Mat92a, Mat92b] for a comprehensive discussion of these matters since our main concern here is with the more general notion of dislocated metric.

3.7 Proposition Any d-ultrametric satisfies (Pii) and (Piv), but not necessarily (Pi).

Proof: Let (X, ϱ) be a d-ultrametric space and let $x, y, z \in X$.
(Pii) By the strong triangle inequality, we obtain $\varrho(x, x) \le \max\{\varrho(x, y), \varrho(y, x)\}$ and by symmetry we obtain the desired inequality.
(Piv) By the strong triangle inequality, we obtain $\varrho(x, z) \le \max\{\varrho(x, y), \varrho(y, z)\}$. Without loss of generality, we can assume that $\varrho(x, y) \ge \varrho(y, z)$. Since by (Pii) we have $\varrho(y, y) \le \varrho(y, z)$, we obtain $\varrho(x, z) \le \varrho(x, y) \le \varrho(x, y) + \varrho(y, z) - \varrho(y, y)$.

Let X be a set and define ϱ on $X \times X$ to be identically 1. Then ϱ is a d-ultrametric on X which does not satisfy (Pi). ■

3.2 Convergence and Continuity

Once the notion of d-nbhood is defined, it is straightforward to adapt the notion of convergence to d-topological spaces.

3.8 Definition Let $(X, \mathcal{U}, \lessdot)$ be a d-topological space and let $x \in X$. A (topological) net (x_λ) *d-converges to* $x \in X$ if for each d-nbhood U of x we have that x_λ is *eventually in* U, that is, there exists some λ_0 such that $x_\lambda \in U$ for each $\lambda > \lambda_0$.

Note that if for some $x \in X$ we have $\emptyset \in \mathcal{U}_x$, then the constant sequence (x) does not d-converge. In fact, if $\emptyset \in \mathcal{U}_x$, then no net in X d-converges to x. Note also that the notion of convergence obtained in Definition 3.8 is a natural generalization of convergence with respect to a d-metric, and we investigate this next.

3.9 Proposition Let (X, ϱ) be a d-metric space and let $(X, \mathcal{U}, \lessdot)$ be the d-topological space obtained from it via the construction in the proof of Proposition 3.5. Let (x_n) be a sequence in X. Then (x_n) converges in ϱ if and only if (x_n) d-converges in $(X, \mathcal{U}, \lessdot)$.

Proof: Let (x_n) be convergent in ϱ to some $x \in X$, so that $\varrho(x_n, x) \to 0$ as $n \to \infty$, and let U be a d-nbhood of x. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Since $\varrho(x_n, x) \to 0$, there exists n_0 such that $x_n \in B \subseteq U$ for all $n > n_0$ and hence (x_n) d-converges to x.

Conversely, let (x_n) be d-convergent to some $x \in X$, that is, for each d-nbhood U of x there exists n_0 such that $x_n \in U$ for each $n > n_0$. For each $\varepsilon > 0$, $B_\varepsilon(x)$ is a d-nbhood of x. Since ε can be chosen arbitrarily small, we must have $\varrho(x_n, x) \to 0$ for $n \to \infty$, as required. ■

We proceed with defining continuity on d-topological spaces.

3.10 Definition Let X and Y be d-topological spaces and let $f : X \to Y$ be a function. Then f is *d-continuous at* $x_0 \in X$ if for each d-nbhood V of $f(x_0)$ in Y there is a d-nbhood U of x_0 in X such that $f(U) \subseteq V$. We say f is *d-continuous on* X if f is d-continuous at each $x_0 \in X$.

The following theorem shows that the notion of d-convergence can be characterized via nets, by analogy with conventional topology.

3.11 Theorem Let X and Y be d-topological spaces and let $f : X \to Y$ be a function. Then f is d-continuous if and only if for each net (x_λ) in X which d-converges to some $x_0 \in X$, $(f(x_\lambda))$ is a net in Y which d-converges to $f(x_0) \in Y$.

Proof: Let f be d-continuous at x_0 and let x_λ be a net which d-converges to x_0. Let V be a d-nbhood of $f(x_0)$. Then there exists a d-nbhood U of x_0 such that $f(U) \subseteq V$. Since x_λ is eventually in U, we obtain that $f(x_\lambda)$ is eventually in V, and hence $f(x_\lambda)$ d-converges to $f(x_0)$.

Conversely, if f is not d-continuous at x_0, then for some d-nbhood V of $f(x_0)$ and for all $U \in \mathcal{U}_{x_0}$ we have $f(U) \nsubseteq V$. Thus for each $U \in \mathcal{U}_{x_0}$ there is an $x_U \in U$ with $f(x_U) \notin V$. Then (x_U) is a net in X which d-converges to x_0 whilst $f(x_U)$ does not d-converge to $f(x_0)$. ■

4 Dislocated Metrics

4.1 Continuity

In Section 3, we have generalized convergence from d-metrics to d-topologies. However, we still lack a notion of continuity for d-metrics. We will investigate this next, and this will enable us to give a proof of Theorem 2.7 which is analogous to the standard proof of the Banach contraction mapping theorem.

4.1 Proposition Let (X, ϱ) and (Y, ϱ') be d-metric spaces, let $f : X \to Y$ be a function and let $(X, \mathcal{U}, \lessdot)$ and $(Y, \mathcal{V}, \lessdot')$ be the d-topological spaces optained from (X, ϱ), respectively (Y, ϱ'), via the construction in Proposition 3.5. Then f is d-continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.

Proof: Let f be d-continuous at $x_0 \in X$ and let $\varepsilon > 0$. Then $B_\varepsilon(f(x_0))$ is a d-nbhood of $f(x_0)$. By definition of d-continuity, there exists a d-nbhood U of x_0 with $f(U) \subseteq B_\varepsilon(f(x_0))$. But since U is a d-nbhood of x_0, there exists a ball $B_\delta(x_0) \subseteq U$ and therefore $f(B_\delta(x_0)) \subseteq f(U) \subseteq B_\varepsilon(f(x_0))$.

Conversely, assume that the ε-δ-condition on f holds and let V be a d-nbhood of $f(x_0)$. Then there exists $\varepsilon > 0$ with $B_\varepsilon(f(x_0)) \subseteq V$ and $\delta > 0$ with $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0)) \subseteq V$. Since $B_\delta(x_0)$ is a d-nbhood of x_0 we obtain d-continuity of f. ■

4.2 Proposition Let (X, ϱ) be a d-metric space, let $f : X \to X$ be a contraction and let $(X, \mathcal{U}, \lessdot\ )$ be the d-topological space optained from (X, ϱ) via the construction in the proof of Proposition 3.5. Then f is d-continuous.

Proof: Let $x_0 \in X$ and let $\varepsilon > 0$ be arbitrarily chosen. For $\delta = \frac{\varepsilon}{\lambda+1}$, we obtain $d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda \frac{\varepsilon}{\lambda+1} < \varepsilon$ for all $x \in B_\delta(x_0)$, and therefore $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$ as required. ■

Proof of Theorem 2.7: With our preparations, the proof follows the proof of the Banach contraction mapping theorem on metric spaces, and we only sketch the details here.

Let $x \in X$ be arbitrarily chosen. Then the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and converges in (X, ϱ) to some point y. Since f is a contraction, it is also d-continuous by Proposition 4.2 from which we obtain $y = \lim f^n(x) = f(\lim f^{n-1}(x)) = f(y)$ by Theorem 3.11. Uniqueness follows since if z is a fixed point of f, then $\varrho(x, z) = \varrho(f(x), f(z)) \leq \lambda\varrho(x, z)$ and therefore $\varrho(x, z) = 0$, and hence $x = z$ by (Mii). ■

It is a corollary of the proof just given that iterates of any point converge to the unique fixed point of the function in question. In denotational semantics, this additional feature of the fixed-point theorem is desirable since it yields a method of actually obtaining the fixed point whose existence has been shown. In particular, from the discussion in Section 2.2 we now obtain that the unique supported model for any acceptable program P can be obtained as the limit of iterates of T_P for any starting interpretation. Such a property might also prove to be useful for studying relationships between logic programs and recurrent neural networks, as in [HSK99], and this is under investigation by the authors.

4.2 Metrics and d-Metrics

In the remaining section, we will investigate relationships between conventional metrics and d-metrics. First note that if f is a contraction on a d-metric X, we have $\varrho(f(x), f(x)) \leq \lambda\varrho(x, x)$ for all $x \in X$. Since the requirement $\varrho(x, x) = 0$ for all $x \in X$ renders a d-metric to be a metric, we are interested in understanding the function $u_\varrho : X \to \mathbb{R}$ defined by $u_\varrho(x) = \varrho(x, x)$.

4.3 Definition Let (X, ϱ) be a d-metric space. The function $u_\varrho : X \to \mathbb{R} : x \mapsto \varrho(x, x)$ is called the *dislocation function* of ϱ.

4.4 Lemma Let (X, ϱ) be a d-metric space. Then $u_\varrho : X \to \mathbb{R}$ is d-continuous.

Proof: Recalling the observations following Definition 3.8, let $x \in X$ and let (x_λ) be a net in X which d-converges to x, that is, for each $\varepsilon > 0$ there exist λ_0 such that $\varrho(x_\lambda, x) < \varepsilon$ for all $\lambda > \lambda_0$. Since $u_\varrho(x_\lambda) = \varrho(x_\lambda, x_\lambda) \leq 2\varrho(x_\lambda, x)$ for all λ, we obtain $u_\varrho(x_\lambda) \to 0$ for increasing λ. It remains to show that $u_\varrho(x) = 0$, and this follows from $u_\varrho(x) = \varrho(x, x) \leq 2\varrho(x_\lambda, x)$, since the latter term tends to 0 for increasing λ. ■

The following is a general result which shows how d-metrics can be obtained from conventional metrics.

4.5 Proposition Let (X, d) be a metric space, let $u : X \to \mathbb{R}_0^+$ be a function and let $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be a symmetric operator which satisfies the triangle inequality. Then (X, ϱ) with

$$\varrho(x, y) := d(x, y) + T(u(x), u(y))$$

is a d-metric space and $u_\varrho(x) = T(u(x), u(x))$ for all $x \in X$. In particular, if $T(x, x) = x$ for all $x \in \mathbb{R}$, then $u_\varrho \equiv u$.

Proof: (Mii) If $\varrho(x, y) = 0$, then $d(x, y) + T(u(x), u(y)) = 0$. Hence $d(x, y) = 0$ and $x = y$.
(Miii) Obvious by symmetry of d and T.
(Miv) Obvious since d and T satisfy the triangle inequality. ■

Completeness also carries over if some continuity conditions are imposed.

4.6 Proposition Using the notation of Proposition 4.5, let u be continuous as a function from (X, d) to $\mathbb{R}_0^+$ (endowed with the usual topology), and let T be continuous as a function from the topological product space $\mathbb{R}^2$ to $\mathbb{R}_0^+$, satisfying the additional property $T(x, x) = x$ for all x. If (X, d) is a complete metric space, then (X, ϱ) is a complete d-metric space.

Proof: Let (x_n) be a Cauchy sequence in (X, ϱ). Thus, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $d(x_m, x_n) \leq d(x_m, x_n) + T(u(x_m), u(x_n)) = \varrho(x_m, x_n) < \varepsilon$. So (x_n) is also a Cauchy sequence in (X, d) and therefore has a unique limit x in (X, d). In particular, we have $x_n \to x$ in (X, d) and also $u(x_n) \to u(x)$ and $T(u(x_n), u(x)) \to T(u(x), u(x)) = u(x)$. We have to show that $\varrho(x_n, x)$ converges to 0 as $n \to \infty$. For all $n \in \mathbb{N}$ we obtain $\varrho(x_n, x) = d(x_n, x) + T(u(x_n), u(x)) \to u(x) = u_\varrho(x)$, and it remains to show that $\varrho(x, x) = 0$. But this follows from the fact that (x_n) is a Cauchy sequence, since this implies that $u(x_n) = u_\varrho(x_n) = \varrho(x_n, x_n) \to 0$ as $n \to \infty$, hence by continuity of u we obtain $u(x) = 0$. ■

We can also obtain a partial converse of Proposition 4.5.

4.7 Proposition Let (X, ϱ) be a d-metric space which satisfies condition (Piv) and let $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be a symmetric operator such that $T(x, x) = x$ for all $x \in \mathbb{R}$ and which satisfies the inequality

$$T(x, y) \geq T(x, z) + T(z, y) - T(z, z)$$

for all $x, y, z \in \mathbb{R}$. Then (X, d) with

$$d(x,y) := \varrho(x,y) - T\left(u_\varrho(x), u_\varrho(y)\right)$$

is a pseudo-metric space.

Proof: (Mi) For all $x \in X$ we have $d(x,x) = \varrho(x,x) - u_\varrho(x) = 0$.
(Miii) Obvious by symmetry of ϱ and T.
(Miv) For all $x, y \in X$ we obtain

$$\begin{aligned} d(x,y) &= \varrho(x,y) - T(u_\varrho(x), u_\varrho(y)) \\ &\leq \varrho(x,z) + \varrho(z,y) - \varrho(z,z) - (T(u_\varrho(x), u_\varrho(z)) + T(u_\varrho(z), u_\varrho(y)) - u_\varrho(z)) \\ &= \varrho(x,z) - T(u_\varrho(x), u_\varrho(z)) + \varrho(z,y) - T(u_\varrho(z), u_\varrho(y)) \\ &= d(x,z) + d(z,y) \end{aligned}$$

■

An example of a natural operator T which satisfies the requirements of Propositions 4.5, 4.6 and 4.7 is

$$T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x,y) \mapsto \frac{1}{2}(x+y).$$

We discuss a few more examples of d-metrics which are partly taken from [Mat92b].

4.8 Example Let d be the metric $d(x,y) = \frac{1}{2}|x-y|$ on $\mathbb{R}_0^+$, let $u : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the identity function, and define $T(x,y) = \frac{1}{2}(x+y)$. Then ϱ as defined in Proposition 4.5 is a d-metric and $\varrho(x,y) = \frac{1}{2}|x-y| + \frac{1}{2}(x+y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_0^+$.

4.9 Example Let $\mathcal{I}$ be the set of all closed intervals on $\mathbb{R}$. Then $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}_0^+$ defined by

$$d([a,b],[c,d]) = \frac{1}{2}\left(|a-c| + |b-d|\right)$$

is a metric on $\mathcal{I}$. Let $u : \mathcal{I} \to \mathbb{R}^+$ be defined by

$$u([a,b]) = b - a$$

and let T be defined as in Example 4.8. Then the construction in the proof of Proposition 4.5 yields a d-metric ϱ such that

$$\varrho([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$$

for all $[a,b],[c,d] \in \mathcal{I}$.
Indeed, we obtain

$$\begin{aligned} \varrho([a,b],[c,d]) &= d([a,b],[c,d]) + \frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}d - \frac{1}{2}c \\ &= \frac{1}{2}\left(|b-d| + b + d + |a-c| - a - c\right) \\ &= \frac{1}{2}\left(|b-d| + (b+d)\right) + \frac{1}{2}\left(|a-c| - (a+c)\right) \\ &= \max\{b,d\} - \min\{a,c\}. \end{aligned}$$

4.10 Example $(\mathbb{R}_0^+, \varrho)$ where $\varrho : (x, y) \mapsto x + y$ is a dislocated metric space.

The following proposition gives an alternative way of obtaining d-ultrametrics from ultrametrics. This result is of importance in the area of denotational semantics where ultrametric structures naturally appear, and the interested reader will note that we have already applied it in Section 2.2.

4.11 Proposition Let (X, d) be an ultrametric space and let $u : X \to \mathbb{R}_0^+$ be a function. Then (X, ϱ) with

$$\varrho(x, y) = \max\{d(x, y), u(x), u(y)\}$$

is a d-ultrametric and $\varrho(x, x) = u(x)$ for all $x \in X$. If u is continuous as a function from (X, d), then completeness of (X, d) implies completeness of (X, ϱ).

Proof: (Mii) and (Miii) are obvious.
(Miv$'$) We obtain

$$\begin{aligned}\varrho(x, y) &= \max\{d(x, y), u(x), u(y)\}\\ &\le \max\{d(x, z), d(z, y), u(x), u(y)\}\\ &\le \max\{d(x, z), u(x), u(z), d(z, y), u(y)\}\\ &\le \max\{\varrho(x, z), \varrho(z, y)\}.\end{aligned}$$

For completeness, let (x_n) be a Cauchy sequence in (X, ϱ). Then (x_n) is a Cauchy sequence in (X, d) and converges to some $x \in X$. We then obtain $\varrho(x_n, x) = \max\{d(x_n, x), u(x_n), u(x)\} \to u(x)$ for $n \to \infty$. As in the proof of Proposition 4.6 we obtain $u(x) = 0$ which completes the proof. ∎

Proposition 4.11 yields that the function ϱ from Section 2.2 is a complete d-ultrametric provided we are able to show that the function u as given there is continuous.

4.12 Lemma Using the notation of Section 2.2, the function $u : I_P \to \mathbb{R}$ defined by $u(K) = \max\{f(K'), d(K', I)\}$ is continuous as a function from (I_P, d) to $\mathbb{R}$.

Proof: Let K_m be a sequence in I_P which converges in d to some $K \in I_P$. We need to show that $d(K'_m, I)$ converges to $d(K', I)$ and $f(K'_m)$ converges to $f(K')$ for $m \to \infty$. Since (K_m) converges to K with respect to the metric d, it follows that for each $n \in \mathbb{N}$ there is $m_n \in \mathbb{N}$ such that K and K_m, for all $m \ge m_n$, agree on all atoms of level less than or equal to n. So if $f(K) = 2^{-n_0}$, say, that means that K'_m and K' agree on all atoms of level less than or equal to n if $m \ge m_{n_0}$, and hence $f(K_m) = f(K)$ for all $m \ge m_{n_0}$. Also, if $d(K', I) = 2^{-n_0}$, say, then $d(K'_m, I) = d(K', I)$ for all $m \ge m_{n_0}$ as required. ∎

5 Discussion

We have studied dislocated topological spaces and an underlying generalized notion of topology, the dislocated topology. Whilst a few applications of dislocated metrics, and in particular of the generalized Banach contraction mapping theorem, Theorem 2.7, are known in Theoretical Computer Science, it is at this stage unclear (and worth investigating) whether or not other applications can be found and where else in Mathematics these spaces appear. The authors are currently developing applications of Theorem 2.7 to logic programs which are more general than acceptable programs. It may also be possible to merge Theorem 2.7 and the fixed point theorem given in [PR00]. This is also under investigation.

The application in Section 2.2 hints at interpreting the dislocation function u_ϱ as a measure of *undesirability*. The interpretation I from the definition of acceptable program could be understood as a first approximation to the desired model, and for each interpretation J the value $u_\varrho(J)$ would be a quantitative evaluation of the desirability of J with respect to I. Whether or not this point of view can be carried over to other settings remains to be seen.

Whilst we have been able to carry over the notions of neighbourhood, convergence and continuity from conventional topologies to dislocated topologies in a way which corresponds to the relationships between these notions in elementary topology, the notion of open set seems to be difficult to recover. On the other hand, with our knowledge of dislocated neighbourhoods it appears to be straightforward to define, for example, dislocated uniformities. If applications can be found in the future, further investigations in these directions will be worth pursuing.

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