Formal Concept Analysis and Resolution in Algebraic Domains — Preliminary Report

Pascal Hitzler and Matthias Wendt

Artificial Intelligence Institute, Department of Computer Science
Dresden University of Technology, Dresden, Germany
{phitzler,mw177754}@inf.tu-dresden.de

Abstract. We relate two formerly independent areas: Formal concept analysis and logic of domains. We will establish a correspondene between contextual attribute logic on formal contexts resp. concept lattices and a clausal logic on coherent algebraic cpos. We show how to identify the notion of formal concept in the domain theoretic setting. In particular, we show that a special instance of the resolution rule from the domain logic coincides with the concept closure operator from formal concept analysis. The results shed light on the use of contexts and domains for knowledge representation and reasoning purposes.

1 Introduction

Domain theory was introduced in the 1970s by Scott as a foundation for programming semantics. It provides an abstract model of computation using order structures and topology, and has grown into a respected field on the borderline between Mathematics and Computer Science [1]. Relationships between domain theory and logic were noted early on by Scott [2], and subsequently developed by many authors, including Smyth [3], Abramsky [4], and Zhang [5]. There has been much work on the use of domain logics as logics of types and of program correctness, with a focus on functional and imperative languages.

However, there has been only little work relating domain theory to logical aspects of knowledge representation and reasoning in artificial intelligence. Two exceptions were the application of methods from quantitative domain theory to the semantic analysis of logic programming paradigms studied by Hitzler and Seda [6, 7], and the work of Rounds and Zhang on the use of domain logics for disjunctive logic programming and default reasoning [8–10]. The latter authors developed a notion of clausal logic in coherent algebraic domains, for convenience henceforth called *logic RZ*, based on considerations concerning the Smyth powerdomain, and extended it to a disjunctive logic programming paradigm [10]. A notion of default negation, in the spirit of answer set programming [11] and Reiter's default logic [12], was also added [13].

The notion of *formal concept* evolved out of the philosophical theory of concepts. Wille [14] proposed the main ideas which lead to the development of formal concept analysis as a mathematical field [15]. The underlying philosophical rationale is that a concept is determined by its *extent*, i.e. the collection of objects

which fall under this concept, and its *intent*, i.e. the collection of properties or attributes covered by this concept. Thus, a formal concept is usually distilled out of an incidence relation between a set of objects and a set of attributes via some concept closure operator, see Section 2 for details. The set of all concepts is then a complete lattice under some natural order, called a *concept lattice*.

The concept closure operator can naturally be represented by an implicational theory of attributes, e.g. the attribute "is a dog" would imply the attribute "is a mammal", to give a simple example. Thus, contexts and concepts determine logical structures, which are investigated e.g. in [16–18]. In this paper, we establish a close relationship between the logical consequence relation in the logic RZ and the construction of concepts from contexts via the mentioned concept closure operator. We will show that finite contexts can be mapped naturally to certain partial orders such that the concept closure operator coincides with a special instance of a resolution rule in the logic RZ, and that the concept lattice of the given context arises as a certain set of logically closed theories. Conversely, we will see how the logic RZ on finite pointed posets finds a natural representation as a context. Finally, we will also see how the contextual attribute logic due to Ganter and Wille [16] reappears in our setting.

Due to the natural capabilities of contexts and concepts for knowledge representation, and the studies by Rounds and Zhang on the relevance of the logic RZ for reasoning mentioned above, the result shows the potential of using domain logics for knowledge representation and reasoning. As such, the paper is part of our investigations concerning the use of domain theory in artificial intelligence, where domains shall be used for knowledge representation, and domain logic for reasoning. The contribution of this paper is on the knowledge representation aspect, more precisely on using domains for representing knowledge which is implicit in formal contexts. Aspects of reasoning, building on the clausal logic of Rounds and Zhang and its extensions, as mentioned above, are being pursued and will be presented elsewhere, and some general considerations can be found in the conclusions. We also note that our results may make way for the use of formal concept analysis for domain-theoretic program analysis, and this issue is also to be taken up elsewhere.

The plan of the paper is as follows. In Section 2 we provide preliminaries and notation from lattice theory, formal concept analysis, and domain theory, which will be needed throughout the paper. In Section 3, we will identify certain logically closed theories from the logic RZ, called *singleton-generated theories*, and show that the set of all these coincides with the Dedekind-MacNeille completion of the underlying finite poset. This sets the stage for the central Section 4 where we will present the main results on the correspondence between concept closure and logical consequence in the logic RZ, as mentioned above. In Section 5 we shortly exhibit how the contextual attribute logic relates to our setting. Finally, in Section 6, we conclude with a general discussion on knowledge representation and reasoning perspectives of our work, and display some of the difficulties involved in carrying over the results to the infinite case. Some parts of the proofs have only been sketched for page limitations.

Acknowledgements. We thank Bernhard Ganter and Guo-Qiang Zhang for inspiring discussions, and Pawel Waszkiewicz for valuable feedback on an earlier version of this paper.

2 Preliminaries and Notation

2.1 Lattice Theory

Our general reference for lattice theory is [19]. A *preorder* is a pair $(P, \leq)$, where $\leq$ is a reflexive and transitive binary relation on P. A *partially ordered set* (*poset*) is a preorder, where $\leq$ is also antisymmetric. A poset $(P, \leq)$ is called *finite* if so is P. We call a poset *pointed* if it has a least, or *bottom* element, i.e. an element $\bot$ with $\bot \leq x$ for all $x \in P$. A *lattice* is a poset such that for any two elements a, b of the lattice there always exists an infimum $a \wedge b$, called the *meet* of a and b, and a supremum $a \vee b$, called the *join* of a and b. A lattice is said to be *complete* if moreover meet and join of any collection of elements exist.

For a poset P, we denote by $\uparrow X = \{y \mid x \leq y \text{ for some } x \in X\}$ the *upper closure* of X, and by $X^\uparrow = \{y \mid x \leq y \text{ for all } x \in X\}$ the (set of) *upper bounds* of X, and for singleton sets $X = \{x\}$ we write $\uparrow x$ and $x^\uparrow$ (which in this case coincide) for simplicity. The notions of *lower closure* and *lower bounds* are obtained dually. A subset $P \subseteq L$ of a lattice L is called *join-dense* in L if each element of L is the join of some elements from P. For a poset P and an embedding $f : P \to L$, we call f a *join-dense embedding* if the image of P under f is join-dense in L. An element $x \in L$ of a lattice L is called *join-irreducible* if x cannot be obtained as the join of two elments distinct from x. The notions of *meet-density* and *meet-irreducibility* are obtained dually. For a given poset P, a pair (X, Y) of subsets of P is called a (*Dedekind*) *cut* in P if $X = Y^\downarrow$ and $Y = X^\uparrow$. The set of all cuts in a poset P, ordered by subset inclusion in the first argument — or equivalently by reverse subset inclusion in the second — is called the *Dedekind-MacNeille completion* of P, denoted by $\mathcal{N}(P)$. It is the smallest complete lattice P can be embedded into, namely by the *principal ideal embedding* $\iota : P \to \mathcal{N}(P) : x \mapsto (\downarrow x, \uparrow x)$, which preserves existing joins and meets. Since any cut $(X, Y) \in \mathcal{N}(P)$ is determined by X, we will for convenience write $X \in \mathcal{N}(P)$, so that e.g. the principal ideal embedding takes the form $x \mapsto \downarrow x$. Note also that for any complete lattice L and any set P which embeds both join- and meet-densely into L we have that $\mathcal{N}(P)$ is isomorphic to L.

2.2 Formal Contexts and Concepts

We introduce basic notions from formal concept analysis, following the standard reference [15].

A (*formal*) *context* is a triple (G, M, I) consisting of two sets G and M and a relation $I \subseteq G \times M$. The elements of G are called the *objects* and the elements of M are called the *attributes* of the context. For $g \in G$ and $m \in M$ we write gIm for $(g, m) \in I$, and say that g *has the attribute* m.

For a set $A \subseteq G$ of objects we set $A' = \{m \in M \mid gIm \text{ for all } g \in A\}$, and for a set $B \subseteq M$ of attributes we set $B' = \{g \in G \mid gIm \text{ for all } m \in B\}$. A (*formal*) *concept* of (G, M, I) is a pair (A, B) with $A \subseteq G$ and $B \subseteq M$, such that $A' = B$ and $B' = A$. We call A the *extent* and B the *intent* of the concept (A, B). For singleton sets, i.e. $B = \{b\}$, we simplify notation by writing b' instead of $\{b\}'$.

The set of all concepts of a given context (G, M, I), ordered by $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$, which is equivalent to the condition $B_2 \subseteq B_1$, is a complete lattice and is denoted by $\underline{\mathfrak{B}}(G, M, I)$. It is called the *concept lattice* of the context (G, M, I), and the following theorem holds, which is a part of the so-called *Basic Theorem* of formal concept analysis.

Theorem 1. *A complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\overline{\gamma} : G \to L$ and $\overline{\mu} : M \to L$ such that $\overline{\gamma}(G)$ is join-dense and $\overline{\mu}(M)$ is meet-dense in L and gIm is equivalent to $\overline{\gamma}(g) \leq \overline{\mu}(m)$.*

In particular, we note that in the setting of Theorem 1 we have that L is isomorphic to the Dedekind-MacNeille completion of the subposet $\overline{\mu}(M) \cup \overline{\gamma}(G)$ of L.

Finally, an object g is called *irreducible* if the respective *object concept* (g'', g') is join-irreducible in $\underline{\mathfrak{B}}(G, M, I)$. Dually, an attribute m is called *irreducible* if the respective *attribute concept* (m', m'') is meet-irreducible in $\underline{\mathfrak{B}}(G, M, I)$. On finite contexts, we have that for every reducible object g, i.e. one which is not irreducible, there exists a set of irreducible objects H such that $H' = g'$, and likewise for attributes.

2.3 The Logic RZ

In [10], Rounds and Zhang developed a clausal logic on certain partial orders which allows a logical characterization of a standard domain-theoretic notion, namely of the Smyth powerdomain of these posets. They also noted, and studied, that their clausal logic, which we call the *logic RZ*, bears potential for establishing a disjunctive logic programming paradigm based on a domain-theoretic background. We will next define those notions from their work which we will use in the sequel. Our discussion, however, will mostly be restricted to the case of finite pointed posets instead of the more general *coherent algebraic cpos* which provide the original setting for the logic RZ. A short discussion of this is deferred to Section 6.

Definition 1. *Let $(P, \sqsubseteq)$ be a finite pointed poset. A* clause *over P is a subset of P. A* theory *over P is a set of clauses over P. For a clause X and $m \in P$, we say that m is a* model *of X, written $m \models X$, if there exists some $x \in X$ with $x \sqsubseteq m$. For a theory T and $m \in P$, we set $m \models T$ if $m \models X$ for all $X \in T$, in which case we call m a* model *of T. For a theory T and a clause X we say that X is a* logical consequence *of T, written $T \models X$, if for all $m \in P$ we have that $m \models T$ implies $m \models X$. A theory T is said to be* logically closed *if $T \models X$ implies $X \in T$ for all clauses X. Given a theory S, we say that T is the* logical closure *of S if T is the smallest logically closed theory containing S. A theory is called* consistent *if it does not have the empty clause as a logical consequence.*

For a theory T, we will denote the set of models of T by $\mathrm{Mod}(T)$. Similarly, given a set $M \subseteq P$ of models, we define the corresponding theory $\mathrm{Th}(M)$ to be the set of all clauses which have all elements of M as model. Note that for every $M \subseteq P$ the corresponding theory $\mathrm{Th}(M)$ contains the clause $\{\bot\}$, hence is non-empty. A central result from [10] is that the set of all consistent and logically closed theories over P, under subset inclusion, is isomorphic to the Smyth powerdomain of P.

3 Singleton-Generated Theories and Poset Completion

In this section, we will show a strong relationship between the logic RZ and poset completion. In particular we show that a set of certain theories is isomorphic to the Dedekind-MacNeille completion of the given poset. Due to the strong link between concept lattices and the Dedekind-MacNeille completion exhibited by Theorem 1, this will provide the necessary tool for our main results, presented in Section 4.

The intuition underlying our observations is that (e.g. classical) logic gives rise to a natural Galois connection between models and theories. In the setting of the logic RZ, this Galois connection arises by representing the Smyth powerdomain of P, together with the inconsistent closed theory, as the concept lattice of the formal context $(P, \mathfrak{C}, \models)$, where $\mathfrak{C}$ is the set of all clauses over P — formal concepts over this context are pairs (M, T) of sets of models and logically closed theories such that $m \models T$ if and only if $m \in M$. Since theories can be thought of as conjunctions of clauses, this yields a Galois connection between conjunctions and their sets of models. Moreover, from the fact that in our special setting elements of clauses and models of theories are drawn from the same set P it will turn out that the Galois connection between conjunctions and their models arises from the formal context $(P, \mathfrak{S}, \models)$, where $\mathfrak{S}$ denotes the set of all singleton clauses over P. This idea will be pursued in detail below.

So we will now look at the analogon to *conjunction* in the logic RZ, which is provided by (finite) theories containing only singleton clauses. Given a finite pointed poset P we will call a logically closed theory T over P *singleton-generated* if and only if there exists a set $M = \{\{d_1\}, \ldots, \{d_n\}\}$, where $n \in \mathbb{N}$ and $d_i \in P$ for each i, such that T is the logical closure of M, and in this case we call $\{d_1, \ldots, d_n\}$ a *generator* of T. For any given logically closed theory S we also set $\mathcal{G}(S) = \{d \mid \{d\} \in S\}$, and note that $\mathcal{G}(S)$ is a generator of S if and only if S is singleton-generated. This definition has been developed out of a closure operator defined in [20]. Now the following theorem can be established.

Theorem 2. *Let P be a finite pointed poset. Then the following hold.*

(i) A logically closed theory T over P is singleton-generated with generator G if and only if $\mathrm{Mod}(T) = G^{\uparrow}$.
(ii) $\big(\mathcal{G}(T), \mathcal{G}(T)^{\uparrow}\big)$ is a cut in P for every logically closed theory T.
(iii) For every cut (X, Y) in P we have that $\mathrm{Th}(Y)$ is singleton-generated and $\mathcal{G}(\mathrm{Th}(Y)) = X$.

(iv) The set of all singleton-generated theories of a finite pointed poset P, ordered by subset inclusion, is isomorphic to the Dedekind-MacNeille completion $\mathcal{N}(P)$ of P.

Proof. First note that given a logically closed theory T we trivially have $m \models \{d\}$ for all $m \in \mathrm{Mod}(T)$ and all $\{d\} \in T$, hence $\mathrm{Mod}(T) \subseteq \mathcal{G}(T)^\uparrow$.

(i) If T is singleton generated, then we additionally have that every upper bound of $\mathcal{G}(T)$ must be a model of T. Conversely, let $\mathrm{Mod}(T) = G^\uparrow$ for some logically closed theory T and some $G \subseteq P$, and let S be generated by G. Then every model m of S is a model for all $\{d\}$ with $d \in G$, hence $m \in G^\uparrow = \mathrm{Mod}(T)$. Likewise, every model m of T is contained in $G^\uparrow$ and is therefore a model of S. So S and T coincide.

(ii) Let T be logically closed. It suffices to show that $\mathcal{G}(T) = \mathcal{G}(T)^{\uparrow\downarrow}$. By definition of $\models$ and logical closure we have that $\{d\} \in T$ iff $d \in \mathrm{Mod}(T)^\downarrow$. Since $\mathrm{Mod}(T) \subseteq \mathcal{G}(T)^\uparrow$ we obtain $\mathrm{Mod}(T)^\downarrow \supseteq \mathcal{G}(T)^{\uparrow\downarrow}$, hence $\mathcal{G}(T) \supseteq \mathcal{G}(T)^{\uparrow\downarrow}$ and so equality.

(iii) Trivially, $Y \subseteq \mathrm{Mod}(\mathrm{Th}(Y))$. We obtain $Y \supseteq \mathrm{Mod}(\mathrm{Th}(Y))$, and hence equality, since there exists a clause C with $Y = {\uparrow} C$ and $C \in \mathrm{Th}(Y)$, which implies that every $m \in \mathrm{Mod}(\mathrm{Th}(Y))$ satisfies $m \models C$ and therefore $m \in Y$. By $X^\uparrow = Y = \mathrm{Mod}(\mathrm{Th}(Y))$ and because $\mathrm{Th}(Y)$ is logically closed we obtain from (i) that $\mathrm{Th}(Y)$ is singleton-generated with generator X. Furthermore, we have $\{x\} \in \mathrm{Th}(Y)$ if and only if $x \in Y^\downarrow = X$, so $\mathcal{G}(\mathrm{Th}(Y)) = X$.

(iv) For every singleton-generated theory T let $\iota(T) = (\mathcal{G}(T), \mathrm{Mod}(T))$. Using (i), (ii), and (iii) it is easily shown that ι is an isomorphism as required.

Before we make use of Theorem 2 in the next section, let us briefly reflect on what we have achieved so far. Identifying singleton-generated theories with cuts yields the possibility of representing finite lattices — which are always complete — by means of finite pointed posets. From an order-theoretic point of view this idea appears to be rather straightforward. Relating this setting to a logic of domains, however, provides a novel aspect. On the one hand, we now have the possibility to use a restricted form of resolution on ordered sets — as will be explained in Section 4 — in order to represent elements of the corresponding Dedekind-MacNeille completion. On the other hand, we obtain a new perspective on the logic RZ, namely that underlying posets can be interpreted from a knowledge representation point of view. In the next section, we will show how Theorem 2 can be employed for relating the logic RZ to formal concept analysis. The following corollary to Theorem 2 will also be helpful. It follows immediately from part (iv) of Theorem 2 together with the remark on the Dedekind-MacNeille completion provided at the end of Section 2.1, noting that every finite lattice is complete.

Corollary 1. *Let L be a finite lattice. Then for every finite pointed poset P which can be embedded join- and meet-densely into L, the set of all singleton-generated theories over P is isomorphic to L.*

Now every complete lattice is the concept lattice of some formal context. We can thus interpret the elements of the Dedekind-MacNeille completion of

a pointed poset P, which can in turn be identified with singleton-generated theories over P, as concepts in the corresponding concept lattice. This indicates that the logic RZ might be used as a knowledge representation formalism. The details of the relationship between the logic RZ on finite pointed posets and concept lattices will be explained in greater detail in the next section.

4 Representation of Formal Contexts by Finite Posets

We will establish a connection between finite posets and formal contexts, showing that the logic RZ can be understood from a knowledge representation perspective. As we have seen in Theorem 2, the set of all singleton-generated theories over a finite pointed poset P is isomorphic to the Dedekind-MacNeille completion $\mathcal{N}(P)$ of P, which will be the central insight needed for our results. In order to make explicit in which way conceptual knowledge is represented in P, we have to identify objects and attributes within P, as follows.

Definition 2. *Let P be a finite pointed poset. An element $x \in P$ is called an* attribute *if it is not the join of all elements strictly below it, and similarly it is called an* object *if it is not the meet of all elements strictly above it.*

The main motivation behind this definition is as follows: In concept lattices, every join-irreducible element corresponds to some object and every meet-irreducible element corresponds to some attribute. Hence we seek to identify those elements in P which correspond to the meet- resp. join-irreducible elements of $\mathcal{N}(P)$. Considering the principal ideal embedding of P into $\mathcal{N}(P)$, we observe that an element $x \in P$ for which $(\downarrow x, \uparrow x)$ is meet-irreducible in $\mathcal{N}(P)$ cannot be the meet of all elements strictly above it. Dualizing the latter — in order to take care of ordering conventions in the different fields — yields the intuition underlying Definition 2.

From the definition of attribute it follows immediately that any singleton-generated theory over P is completely determined by the set of attributes it contains as singletons: any singleton which is not an attribute can be represented as the join of all the attributes below it, hence is derivable from these attributes in the logic RZ. This is stated formally in the following lemma.

Lemma 1. *Let P be a finite pointed poset and T_1, T_2 be two singleton-generated theories over P which coincide on all attributes of P. Then $T_1 = T_2$.*

Proof. Assume $\{x\} \in T_1$ for some non-attribute $x \in P$. Now x is the join of all the attributes below it, and by logical closure $\{m\} \in T_1$ for all attributes $m \leq x$, hence $\{m\} \in T_2$ for all attributes $m \leq x$. So by logical closure of T_2 we obtain $\{x\} \in T_2$. The argument clearly reverses and therefore suffices.

At this stage our way of interpreting singleton-generated theories as concepts becomes almost obvious. An object $g \in P$ is in the extent of a concept (i.e. of a singleton-generated theory) T if $g \models T$, and an attribute $m \in P$ is in the intent of a concept T if $\{m\} \in T$. The latter means that every object

in the extent of T necessarily has attribute m. Furthermore, if an object g is contained as a singleton in a theory, then it is necessary for any other object in the corresponding concept extent to have all the attributes g has. If an attribute is a model for a theory, then any object having this attribute is also a model of the theory.

When reasoning about the knowledge represented in a poset P, we can — according to Lemma 1 — restrict our attention to the attributes. But we can also incorporate the objects into the reasoning, if desired, as a kind of *macros* for a collection of attributes. This perspective will be employed later on when discussing the logic programming framework developed by Rounds and Zhang [10] in terms of formal concept analysis.

Theorem 3. *Let (G, M, I) be a finite formal context. Then $\underline{\mathfrak{B}}(G, M, I)$, under the reverse order, is isomorphic to the set of all singleton-generated theories of a finite pointed poset P, under subset inclusion, if and only if there exist mappings $\gamma : G_{irr} \to P$ and $\mu : M_{irr} \to P$, where G_{irr} resp. M_{irr} denote the irreducible objects resp. attributes of (G, M, I), such that the following conditions hold.*

(i) $\gamma(G_{irr})$ resp. $\mu(M_{irr})$ contain the objects resp. attributes of P according to Definition 2.
(ii) For all $g \in G_{irr}$ and $m \in M_{irr}$ we have gIm if and only if $\gamma(g) \geq \mu(m)$.

In particular, for a finite pointed poset P with objects G and attributes M, the set of all singleton-generated theories over P under reverse subset inclusion is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$, where I is the restriction of $\geq$ to $G \times M$.

Moreover, for a finite pointed poset P with objects G and attributes M we have that (X, Y) is a concept of the corresponding context if and only if there exists a singleton generated theory T with $X = \mathcal{G}(T) \cap M$ and $Y = \mathrm{Mod}(T) \cap G$.

Proof. For the *if*-direction of the first part, let $(P, \leq)$ be a finite poset and note that any join-irreducible element of $\mathcal{N}(P)$ must be in the image of P under the principal ideal embedding ι, since ι embeds P join- and meet-densely in $\mathcal{N}(P)$.

We first show that whenever $\downarrow m \in \mathcal{N}(P)$ is join-irreducible, then m is an attribute of P. So assume the converse, namely that m is the join of all elements strictly below it. Since the principal ideal embedding ι preserves existing joins and meets, $\downarrow m$ is the join of all $\downarrow x$, where $x < m$, in $\mathcal{N}(P)$. Thus, using finiteness, $\downarrow m$ cannot be join-irreducible in $\mathcal{N}(P)$. Reasoning dually for the objects shows that each meet-irreducible element of $\mathcal{N}(P)$ is the image of an object of P.

We next define a mapping $\overline{\mu} : M \to \mathcal{N}(P)$. For $m \in M_{\mathrm{irr}}$ let $\overline{\mu}(m) = \downarrow \mu(m)$. From the argument just given we obtain that $\overline{\mu}(M_{\mathrm{irr}})$ is join-dense in $\mathcal{N}(P)$. For $m \in M \setminus M_{\mathrm{irr}}$ we have that m is a reducible attribute, i.e. there exists $N \subseteq M_{\mathrm{irr}}$ with $N' = m'$. In this case, let $\overline{\mu}(m) = \bigcup\{\downarrow \mu(n) \mid n \in N\}$, and $\overline{\mu}$ is easily shown to be well-defined. The mapping $\overline{\gamma} : G \to \mathcal{N}(P)$ is obtained dually.

It is now straightforward to verify that $\overline{\gamma}$ and $\overline{\mu}$ satisfy the duals of the conditions from Theorem 1, the application of which yields that $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the Dedekind-MacNeille completion of P under the reverse order.

Conversely, let $\underline{\mathfrak{B}}(G, M, I)$ be isomorphic, under the reverse ordering, to the set of all singleton-generated theories, under subset inclusion, of some finite

pointed poset P. By Theorems 1 and 2 this implies that there are mappings $\overline{\gamma} : G \to \mathcal{N}(P)$ and $\overline{\mu} : M \to \mathcal{N}(P)$ such that $\overline{\gamma}(G)$ is meet-dense and $\overline{\mu}(M)$ is join-dense in $\mathcal{N}(P)$, and such that gIm is equivalent to $\overline{\gamma}(G) \geq \overline{\mu}(M)$. These mappings restrict to mappings from the set of irreducible objects resp. attributes to meet- resp. join-irreducible elements of $\mathcal{N}(P)$. Since P embeds join- and meet-densely into $\mathcal{N}(P)$, every join- resp. meet-irreducible element of $\mathcal{N}(P)$ is the image of an attribute resp. object in P, which proves the claim.

The second part is a direct consequence of the first part, using Theorem 2.

The final statement is a reformulation of Theorem 2, using the observation from Lemma 1 that we can ignore non-attributes when generating theories. A similar argument shows that the sets of models of singleton-generated theories are determined by the objects they contain.

Next, we will give a specific example for a construction of a finite pointed poset from a given context (G, M, I). Note that also $\underline{\mathfrak{B}}(G, M, I)$, reversely ordered, is a finite pointed poset trivially satisfying the conditions from Theorem 3.

Example 1. Let (G, M, I) be a formal context, where G and M are finite and disjoint. Define the following ordering on $G \dot{\cup} M$:

(i) For $m_1, m_2 \in M$ let $m_1 \leq m_2$ if $m_1' \supseteq m_2'$.
(ii) For $g_1, g_2 \in G$ let $g_1 \leq g_2$ if $g_1' \subseteq g_2'$.
(iii) For $g \in G$ and $m \in M$ let $m \leq g$ if gIm.
(iv) For $g \in G$ and $m \in M$ let $g \leq m$ if for all $h \in G$ and all $n \in M$ we have that gIn and hIm imply hIn.[1]

The above construction yields a preorder on $G \dot{\cup} M$. We obtain from this a partial order, also denoted by $\leq$, by taking the quotient order in the usual way. If $(G \dot{\cup} M/_\sim, \leq)$ does not have a least element, we add $\bot$ to $G \dot{\cup} M/_\sim$ and set $\bot \leq x$ for all $x \in G \dot{\cup} M/_\sim$. The latter amounts to adding an additional attribute m with $m' = G$ to the context.

The main intuition behind this construction is to use the set consisting of all objects and attributes as a join- and meet-dense subset of the concept lattice and to supply the induced order by constructions directly available from the formal context. The first three items do exactly this. However, we have to take care of those elements which are both join- and meet-irreducible in the concept lattice. This is achieved with (iv) and the quotient order construction, where those object-attribute pairs are identified which will result in doubly irreducible elements. The following proposition shows that Example 1 is correct.

Proposition 1. *The poset $P = (G \dot{\cup} M/_\sim, \leq)$ as defined in Example 1 satisfies the properties from the first part of Theorem 3.*

Proof. The mapping $\mu : M_{\text{irr}} \to P$ is defined by $\mu(m) = [m]$, where $[m]$ denotes the equivalence class of m under $\sim$. Likewise, set $\gamma : G_{\text{irr}} \to P : g \mapsto [g]$. We need

[1] Bernhard Ganter personal communication.

to verify that μ and γ satisfy the properties from the first part of Theorem 3. The second stated property that gIm if and only if $\gamma(g) \geq \mu(m)$ clearly holds.

So let $[x]$ be an attribute of P according to Definition 2. If $x \in M_{\text{irr}}$, then there is nothing to show. If $x \in M \setminus M_{\text{irr}}$, then x is a reducible attribute, so there exists a set $N \subseteq M_{\text{irr}}$ with $N' = x'$. From $x' = \bigcap_{n \in N} n'$ we obtain $[x] = \bigvee_{n \in N} [n]$, which contradicts that x is an attribute of P. So $x \notin M \setminus M_{\text{irr}}$.

Finally, consider the case $x \in G$. If there is $y \in M_{\text{irr}}$ with $[y] = [x]$ then again there is nothing to show. So assume that this is not the case, which means that by the argument from the preceeding paragraph we have $y \in G$ for all y with $[y] = [x]$. Now from the fact that $[x]$ is an attribute of P we obtain that either there exists $[y] \in P$ such that $[y] \leq [z] < [x]$ implies $[y] = [z]$ for all z, or $\downarrow[x] \setminus \{[x]\}$ has minimal upper bounds different from $[x]$.

In the first case, if $y \in G$ then for every $m \in M$ with xIm we have yIm by (iii), i.e. $x' \subseteq y'$ and by (ii) we obtain $[x] \leq [y]$, hence $[x] = [y]$, which is impossible. If $y \in M$ then let $h \in G$ and $n \in M$ with xIn and hIy. By (iii) and (i) we obtain $[n] \leq [y]$ and by (iii) and (ii) we obtain $[x] \leq [h]$ and hIn by (iii). So $[x] \leq [y]$ by (iv) which again is impossible.

In the second case, let $[y]$ be a minimal upper bound of $\downarrow[x] \setminus \{[x]\}$ different from $[x]$. If $y \in G$ then by (iii) we obtain $x' \subseteq y'$, hence $[x] \leq [y]$ by (ii), which is impossible. If $y \in M$ then let $h \in G$ and $n \in M$ with xIn and hIy. By (iii) and (i) we obtain $[n] \leq [y]$ and by (iii) and (ii) we obtain $[x] \leq [h]$ and hIn by (iii). So $[x] \leq [y]$ by (iv) which again is impossible.

We have just concluded the proof that whenever $[x]$ is an attribute of P then $x \in M_{\text{irr}}$. A similar reasoning shows that whenever $[y]$ is an object in P then $y \in G_{\text{irr}}$. So Theorem 3 is applicable.

Having established Theorem 3 as a link between the logic RZ and formal concept analysis, we will now discuss how the different techniques on both sides embed. In particular, we will shortly consider a proof theory for the logic RZ, discussed next, and also the contextual attribute logic of formal concept analysis, discussed in Section 5.

In [10], the following hyperresolution rule was presented:

$$\frac{X_1\, X_2 \ldots X_n; \quad a_i \in X_i \quad \text{for } 1 \leq i \leq n; \quad \text{mub}\{a_1, \ldots, a_n\} \models Y}{Y \cup \bigcup_{1 \leq i \leq n} (X_i \setminus \{a_i\})}$$

In words, this rule says that from clauses $X_1, \ldots, X_n$, $a_i \in X_i$ for all i, and $\text{mub}\{a_1, \ldots, a_n\} \models Y$ with respect to the logic RZ, the clause $Y \cup \bigcup_{1 \leq i \leq n} (X_i \setminus \{a_i\})$ may be derived. This rule, together with two special rules treating the cases of an empty selection of clauses resp. an empty clause in the premise of the rule, yields a proof theory resp. entailment relation $\vdash$ which is sound and complete w.r.t. the model theory given in Definition 1. From our results, in particular from Theorem 3, we obtain that the following restriction of the hyperresolution rule to singleton clauses induces an entailment relation $\vdash_s$ which is equivalent to the concept closure operator $(\cdot)'' : \mathcal{P}(M) \to \mathcal{P}(M)$ which maps any set of attributes B to the intent B'' of the corresponding concept (B', B''):

$$\frac{\{a_1\}\,\{a_2\}\ldots\{a_n\};\quad \mathrm{mub}\{a_1,\ldots,a_n\} \models \{a\}}{\{a\}}$$

Thus we can conclude that the logic RZ can be used for knowledge representation in much the same way as formal concept analysis: There is a correspondence between finite formal contexts and finite pointed posets and, moreover, both the proof and the model theory of [10] lend themselves to an easy characterization of concept closure. This is probably not too surprising from the viewpoint of formal concept analysis resp. lattice theory. However, from the viewpoint of domain theory it is certainly interesting that there is such a close correspondence between domain logics developed for reasoning about program semantics [4] and a knowledge representation mechanism like formal concept analysis.

5 Contextual Attribute Logic and the Logic RZ

In this section, we will show that the correspondence between the logic RZ and formal concept analysis is not exhausted by the relationship between singleton-generated theories and concept closure. In particular, we will show how to identify part of the contextual attribute logic due to [16] in a finite pointed poset P by means of the logic RZ. We first show how clauses and theories resemble constructions of compound attributes in the poset.

In [16], compound attributes are defined to be compositions of attributes w.r.t. their extent. More precisely, for any set $A \subseteq M$ of attributes of a formal context (G, M, I), the compound attribute $\bigvee A$ has the extent $\bigcup\{m' \mid m \in A\}$, and the compound attribute $\bigwedge A$ has the extent $\bigcap\{m' \mid m \in A\}$. For an attribute $m \in M$, the compound attribute $\neg m$ has the extent $G \setminus m'$.

Now we can relate compound attributes and theories in the logic RZ by the following proposition, which is in fact a straightforward consequence of our previous results, so we skip the proof.

Proposition 2. *Let P be a finite pointed poset and consider the formal context (G, M, I) obtained from it as indicated in Theorem 3, and let γ, μ be as in the same theorem. Then for all $A \subseteq M$, $g \in G$, and $m \in M$ the following hold.*

- *g is in the extent of $\bigvee A$ if and only if $\gamma(g) \models \mu(A)$.*
- *g is in the extent of $\bigwedge A$ if and only if $\gamma(g) \models \{\{\mu(a)\} \mid a \in A\}$.*
- *g is in the extent of $\neg m$ if and only if $\gamma(g) \not\models \{\mu(m)\}$.*

We thus see, that the formation of conjunction and disjunction of attributes to compound attributes corresponds exactly to the formation of singleton generated theories resp. clauses. Negation, however, is more difficult to represent in the logic RZ, since the set of all models of $\neg m$ is not an upper set, but a lower set, more precisely it is the complement of a principal filter in P. Thus it seems that the Scott topology, on which the logic RZ is implicitly based, is not appropriate for handling this kind of negation — which could be a candiate for a strong negation in the logic programming paradigm discussed in Section

6. It remains to be investigated whether the results presented in [10] carry over to the Lawson topology and the Plotkin powerdomain, which according to what has been said above may be the correct setting for handling this negation.

In [16], sequents of the form (A, S), where $A, S \subseteq M$, were introduced as a possible reading of compound attributes $\bigvee(S \cup \{\neg m \mid m \in A\})$. A sequent (A, S) may thus be interpreted as an implication $\bigvee S \leftarrow \bigwedge A$. A *clause set* over M is a set of sequents over M. The clause logic, called *contextual attribute logic*, of a finite pointed poset P is then the set of all sequents that are *all-extensional* in P, i.e. all sequents whose extent contains the set of all objects of P. This means that the implication represented by the sequent holds for all the objects in P.

Due to the difficulties with negation discussed above we restrict our attention, for the time being, to all-extensional sequents (A, S) with $A = \emptyset$. So consider again the setting of Proposition 2, and let $X \subseteq M$. Then $(\emptyset, X)$ is an all-extensional sequent if and only if $\gamma(x) \models \mu(X)$ for all $x \in G$. This is easily verified using Theorem 3 and Proposition 2.

Apart from investigating compound attributes involving negation — as discussed above — it also remains to be determined whether there exists a way of identifying the contextual attribute logic by means of the proof theory defined by Rounds and Zhang [10]. This will be subject to further research.

6 Conclusions and Further Work

We have displayed a strong relationship between formal concept analysis and the domain logic RZ. The restriction of inference to singleton clauses yields the concept closure operator of formal concept analysis. Furthermore, any logically closed theory in the logic RZ can be understood as a clause set over a formal context, in the sense of contextual attribute logic, and the hyperresolution rule of [10] can be used to reason about such knowledge present in a given formal context in much the same way as the resolution rule proposed in [16]. This of course can be a foundation for logic programming over formal contexts, i.e. logic programming with background knowledge which is taken from a formal context and used as "hard constraints".

The appropriate way of doing this on domains was also studied by Rounds and Zhang. In their logic programming paradigm on coherent algebraic cpos a logic program is a set of rules of the form $\theta \leftarrow \tau$, where θ and τ are clauses over the respective domain. The rule

$$\frac{X_1\ X_2 \dots X_n; \quad a_i \in X_i \quad \text{for } 1 \leq i \leq n; \quad \theta \leftarrow \tau \in P; \quad \mathrm{mub}\{a_1, \dots, a_n\} \models \tau}{\theta \cup \bigcup_{1 \leq i \leq n} (X_i \setminus \{a_i\})}$$

corresponds to inference taking the clause $\theta \leftarrow \tau$ into account. By adjoining to the usual proof theory the inference rules for all the clauses in a given program, one can define a monotonic and continuous operator T_P on the set of all logically closed theories, whose least fixed point yields a very satisfactory semantics, i.e. model, for the considered program P.

This logic programming paradigm can be understood as logic programming with background knowledge, since the least model of the program is not only a model for the program, in a reasonable sense, but also takes into account those implications which are hidden in the underlying domain, i.e. in the context. It is interesting to note that the knowledge implicit in the context need not be made explicit, e.g. by computing the stem base of the context. This implicational knowledge is implicitly represented by the inference rules constituting the proof theory of the logic RZ. The authors are currently investigating the potential of this approach.

In this paper, we have restricted our considerations to the case of finite pointed posets. So let us shortly discuss some of the difficulties involved in carrying over our results to the case of arbitrary coherent algebraic cpos. The correspondence between singleton-generated theories and cuts from Theorem 2 carries over to the infinite case without major restrictions — one just has to correctly adjust it to compact elements and to keep in mind that any non-compact element can be represented as the supremum of all compact elements below it. Difficulties occur when trying to characterize the lattices which arise as Dedekind-MacNeille completions of coherent algebraic cpos, since on the domain-theoretic side one has to deal with the topological notion of coherence, which is not really present on the lattice-theoretic side. Furthermore, the Scott-topology we are implicitly dealing with when working with the logic RZ is not completion invariant, which means that the properties defined in terms of the Scott topology, e.g. continuity of the poset, do not carry over to the completion [21]. These issues will also have to be subject to further research. A construction similar to Example 1 carries over to a restricted infinite case, and details can be found in [20].

We finally note the very recent paper by Zhang [22], which also studies relationships between domain theory and formal concept analysis, though from a very different perspective involving Chu spaces.

References

1. Abramsky, S., Jung, A.: Domain theory. In Abramsky, S., Gabbay, D., Maibaum, T.S., eds.: Handbook of Logic in Computer Science. Volume 3. Clarendon, Oxford (1994)
2. Scott, D.S.: Domains for denotational semantics. In Nielsen, M., Schmidt, E.M., eds.: Automata, Languages and Programming, 9th Colloquium, July 1982, Aarhus, Denmark, Proceedings. Volume 140 of Lecture Notes in Computer Science., Springer, Berlin (1982) 577–613
3. Smyth, M.B.: Powerdomains and predicate transformers: A topological view. In Díaz, J., ed.: Automata, Languages and Programming, 10th Colloquium, July 1989, Barcelona, Spain, Proceedings. Volume 298 of Lecture Notes in Computer Science., Springer, Berlin (1989) 662–675
4. Abramsky, S.: Domain theory in logical form. Annals of Pure and Applied Logic **51** (1991) 1–77
5. Zhang, G.Q.: Logic of Domains. Birkhauser, Boston (1991)

6. Hitzler, P.: Generalized Metrics and Topology in Logic Programming Semantics. PhD thesis, Department of Mathematics, National University of Ireland, University College Cork (2001)
7. Hitzler, P., Seda, A.K.: Generalized metrics and uniquely determined logic programs. Theoretical Computer Science (200x) To appear.
8. Zhang, G.Q., Rounds, W.C.: Power defaults (preliminary report). In Dix, J., Furbach, U., Nerode, A., eds.: Proceedings of the Fourth International Conference on Logic Programming and Non-Monotonic Reasoning, LPNMR'97, Dagstuhl, Germany. Volume 1265 of Lecture Notes in Computer Science., Springer (1997) 152–169
9. Zhang, G.Q., Rounds, W.C.: Semantics of logic programs and representation of Smyth powerdomains. In Keimel, K., et al., eds.: Domains and Processes. Kluwer (2001) 151–179
10. Rounds, W.C., Zhang, G.Q.: Clausal logic and logic programming in algebraic domains. Information and Computation **171** (2001) 156–182
11. Marek, V.W., Truszczyński, M.: Stable models and an alternative logic programming paradigm. In Apt, K.R., Marek, V.W., Truszczyński, M., Warren, D.S., eds.: The Logic Programming Paradigm: A 25-Year Persepective. Springer, Berlin (1999) 375–398
12. Reiter, R.: A logic for default reasoning. Artificial Intelligence **13** (1980) 81–132
13. Hitzler, P.: Towards nonmonotonic reasoning on hierarchical knowledge. In: Proceedings of the 17th Workshop Logische Programmierung, WLP02, December 2002, Dresden, Germany. (200x) To appear.
14. Wille, R.: Restructuring lattice theory: An approach based on hierarchies of concepts. In Rival, I., ed.: Ordered Sets. Reidel, Dordrecht-Boston (1982) 445–470
15. Ganter, B., Wille, R.: Formal Concept Analysis — Mathematical Foundations. Springer, Berlin (1999)
16. Ganter, B., Wille, R.: Contextual attribute logic. In Tepfenhart, W.M., Cyre, W.R., eds.: Conceptual Structures: Standards and Practices. Proceedings of the 7th International Conference on Conceptual Structures, ICCS '99, July 1999, Blacksburgh, Virginia, USA. Volume 1640 of Lecture Notes in Artificial Intelligence., Springer, Berlin (1999) 377–388
17. Ganter, B.: Attribute exploration with background knowledge. Theoretical Computer Science **217** (1999) 215–233
18. Wille, R.: Boolean judgement logic. In Delugach, H., Stumme, G., eds.: Conceptual Structures: Broadening the Base, Proceedings of the 9th International Conference on Conceptual Structures, ICCS 2001, July 2001, Stanford, LA, USA. Volume 2120 of Lecture Notes in Artificial Intelligence., Springer, Berlin (2001) 115–128
19. Davey, B., Priestley, H.: Introduction to Lattices and Order. Cambridge University Press (1990)
20. Hitzler, P.: Contexts, concepts and logic of domains. Technical Report WV-02-12, Knowledge Representation and Reasoning Group, Department of Computer Science, Dresden University of Technology, Dresden, Germany (2002)
21. Erné, M.: A completion-invariant extension of the concept of continuous lattices. In Banaschewski, B., Hoffmann, R.E., eds.: Continuous Lattices: Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices. Volume 871 of Lecture Notes in Mathematics., Berlin, Springer (1979) 43–60
22. Zhang, G.Q.: Chu spaces, concept lattices, and domains. In: Proceedings of the Nineteenth Conference on the Mathematical Foundations of Programming Semantics, March 2003, Montreal, Canada. Electronic Notes in Theoretical Computer Science (2003) To appear.