Level Mapping Characterizations of Selector Generated Models for Logic Programs

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Abstract. Assigning semantics to logic programs via selector generated models (Schwarz 2002/2003) extends several semantics, like the stable, the inflationary, and the stable generated semantics, to programs with arbitrary formulae in rule heads and bodies. We study this approach by means of a unifying framework for characterizing different logic programming semantics using level mappings (Hitzler and Wendt 200x, Hitzler 2003), thereby supporting the claim that this framework is very flexible and applicable to very diversely defined semantics.

1 Introduction

Hitzler and Wendt [8–10] have recently proposed a unifying framework for different logic programming semantics. This approach is very flexible and allows to cast semantics of very different origin and style into uniform characterizations using level mappings, i.e. mappings from atoms to ordinals, in the spirit of the definition of acceptable programs [2], the use of stratification [1, 13] and a characterization of stable models by Fages [3]. These characterizations display syntactic and semantic dependencies between language elements by means of the preorders on ground atoms induced by the level mappings, and thus allow inspection of and comparison between different semantics, as exhibited in [8–10].

For the syntactically restricted class of normal logic programs, the most important semantics — and some others — have already been characterized and compared, and this was spelled out in [8–10]. Due to the inherent flexibility of the framework, it is clear that studies of extended syntax are also possible, but have so far not been carried out. In this paper, we will present a non-trivial technical result which provides a first step towards a comprehensive comparative study of different semantics for logic programs under extended syntax.

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Table 1. Notions of specific types of rules.

rule is called	*set*	*condition*
definite	LP	$\mathsf{body}(r) \in \mathsf{Lg}\left(\{\wedge, \mathbf{t}\}, A\right)$ and $\mathsf{head}(r) \in A$
normal	NLP	$\mathsf{body}(r) \in \mathsf{Lg}\left(\{\wedge, \mathbf{t}\}, \mathsf{Lit}\left(A\right)\right)$ and $\mathsf{head}(r) \in A$
head-atomic	HALP	$\mathsf{body}(r) \in \mathsf{Lg}\left(\Sigma^{cl} \setminus \{\mathbf{f}\}, A\right)$ and $\mathsf{head}(r) \in A$
pos. head disj.	DLP^{+}	$\mathsf{body}(r) \in \mathsf{Lg}\left(\{\wedge, \mathbf{t}\}, \mathsf{Lit}\left(A\right)\right)$ and $\mathsf{head}(r) \in \mathsf{Lg}\left(\{\vee\}, A\right)$
disjunctive	DLP	$\mathsf{body}(r) \in \mathsf{Lg}\left(\{\wedge, \mathbf{t}\}, \mathsf{Lit}\left(A\right)\right)$, $\mathsf{head}(r) \in \mathsf{Lg}\left(\{\vee, \mathbf{f}\}, \mathsf{Lit}\left(A\right)\right)$
head-disjunctive	HDLP	$\mathsf{body}(r) \in \mathsf{Lg}\left(\Sigma^{cl} \setminus \{\mathbf{f}\}, A\right)$, $\mathsf{head}(r) \in \mathsf{Lg}\left(\{\vee, \mathbf{f}\}, \mathsf{Lit}\left(A\right)\right)$
generalized	GLP	no condition

More precisely, among the many proposals for semantics for logic programs under extended syntax we will study a very general approach due to Schwarz [14–16]. In this framework, arbitrary formulae are allowed in rule heads and bodies, and it encompasses the inflationary semantics [11], the stable semantics [5], the stable semantics for disjunctive programs [12], and the stable generated semantics [7]. It can itself be understood as a unifying framework for different semantics.

In this paper, we will provide a single theorem — and some corollaries thereof — which gives a characterization of general selector generated models by means of level mappings. It thus provides a link between these two frameworks, and implicitly yields level mapping characterizations of the semantics encompassed by the selector generated approach.

The plan of the paper is as follows. In Section 2 we will fix preliminaries and notation. In Section 3 we will review selector generated models as introduced in [14–16]. In Section 4, we will prove our main result, Theorem 4, which gives a level-mapping characterization of general selector generated models in the style of [8–10]. In Section 5 we study corollaries from Theorem 4 concerning specific cases of interest encompassed by the result. We eventually conclude and discuss further work in Section 6.

2 Preliminaries

Throughout the paper, we will consider a language $\mathcal{L}$ of propositional logic over some set of propositional variables, or *atoms*, A, and connectives $\Sigma^{cl} = \{\neg, \vee, \wedge, \mathbf{t}, \mathbf{f}\}$, as usual. A *rule* r is a pair of formulae from $\mathcal{L}$ denoted by $\varphi \Rightarrow \psi$. φ is called the *body* of the rule, denoted by $\mathsf{body}(r)$, and ψ is called the *head* of the rule, denoted by $\mathsf{head}(r)$. A *program* is a set of rules. A *literal* is an atom or a negated atom, and $\mathsf{Lit}\left(A\right)$ denotes the set of all literals in $\mathcal{L}$. For a set of connectives $C \subseteq \Sigma^{cl}$ we denote by $\mathsf{Lg}\left(C, A\right)$ the set of all formulae over $\mathcal{L}$ in which only connectives from C occur.

Further terminology is introduced in Table 1. The abbreviations in the second column denote the sets of all rules with the corresponding property. A program containing only definite (normal, etc.) rules is called *definite* (*normal*, etc.).

Programs not containing the negation symbol $\neg$ are called *positive*. *Facts* are rules r where $\mathsf{body}(r) = \mathbf{t}$, denoted by $\Rightarrow \mathsf{head}(r)$. Any set B of atoms defines the set of facts $\mathsf{fact}(B) = \{\Rightarrow a \mid a \in B\}$.

The *base* B_P is the set of all atoms occurring in a program P. A two-valued *interpretation* of a program P is represented by a subset of B_P, as usual. By $\mathbf{I}_P$ we denote the set of all interpretations of P. It is a complete lattice with respect to the subset ordering $\subseteq$. For an interpretation $I \in \mathbf{I}_P$, we define $\uparrow I = \{J \in \mathbf{I}_P \mid I \subseteq J\}$ and $\downarrow I = \{J \in \mathbf{I}_P \mid J \subseteq I\}$. $[I, J] = \uparrow I \cap \downarrow J$ is called an *interval* of interpretations.

The model relation $M \models \varphi$ for an interpretation M and a propositional formula φ is defined as usual in propositional logic, and $\mathrm{Mod}(\varphi)$ denotes the set of all models of φ. Two formulae φ and ψ are *logically equivalent*, written $\varphi \equiv \psi$, iff $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$.

A formula φ is *satisfied* by a set $\mathbf{J} \subseteq \mathbf{I}_P$ of interpretations if each interpretation $J \in \mathbf{J}$ is a model of φ. For a program P, a set $\mathbf{J} \subseteq \mathbf{I}_P$ of interpretations determines the set of all rules which *fire* under $\mathbf{J}$, formally $\mathsf{fire}(P, \mathbf{J}) = \{r \in P \mid \forall J \in \mathbf{J} : J \models \mathsf{body}(r)\}$. An interpretation M is called a *model* of a rule r (or *satisfies* r) if M is a model of the formula $\neg\mathsf{body}(r) \vee \mathsf{head}(r)$. An interpretation M is a *model* of a program P if it satisfies each rule in P.

For conjunctions or disjunctions φ of literals, φ^+ denotes the set of all atoms occurring positively in φ, and φ^- contains all atoms that occur negated in φ. For instance, for the formula $\varphi = (a \wedge \neg b \wedge \neg a)$ we have $\varphi^+ = \{a\}$ and $\varphi^- = \{a, b\}$. In heads φ consisting only of disjunctions of literals, we always assume without loss of generality that $\varphi^+ \cap \varphi^- = \emptyset$.

If φ is a *conjunction* of literals, we abbreviate $M \models \bigwedge_{a \in \varphi^+} a$ (i.e. $\varphi^+ \subseteq M$) by $M \models \varphi^+$ and $M \models \bigwedge_{a \in \varphi^-} \neg a$ (i.e. $\varphi^- \cap M = \emptyset$) by $M \models \varphi^-$, abusing notation. If φ is a *disjunction* of literals, we write $M \models \varphi^+$ for $M \models \bigvee_{a \in \varphi^+} a$ (i.e. $M \cap \varphi^+ \neq \emptyset$) and $M \models \varphi^-$ for $M \models \bigvee_{a \in \varphi^-} \neg a$ (i.e. $\varphi^- \not\subseteq M$).

By iterative application of rules from a program $P \subseteq \mathsf{GLP}$ starting in the least interpretation $\emptyset \in \mathbf{I}_P$, we can create monotonically increasing (transfinite) sequences of interpretations of the program P, as follows.

Definition 1. *A (transfinite) sequence C of length α of interpretations of a program $P \subseteq \mathsf{GLP}$ is called a P-*chain* iff*

(C0) $C_0 = \emptyset$,
(Cβ) $C_{\beta+1} \in \mathrm{Min}(\uparrow C_\beta \cap \mathrm{Mod}(\mathsf{head}(Q_\beta)))$ *for some set of rules $Q_\beta \subseteq P$ and for all β with $\beta + 1 < \alpha$, and*
(Cλ) $C_\lambda = \bigcup\{C_\beta \mid \beta < \lambda\}$ *for all limit ordinals $\lambda < \alpha$.*

$\mathbf{C}_P$ *denotes the collection of all P-chains.*

Note that all P-chains increase monotonically with respect to $\subseteq$.

In the proof of Theorem 4, we will make use of the following straightforward lemma from [16].

Lemma 1. *For any set of interpretations $\mathbf{J} \subseteq \mathbf{I}_P$ and any interpretation $K \in \mathbf{I}_P$ we have* $\mathrm{Min}(\mathbf{J} \cap \downarrow K) = \mathrm{Min}(\mathbf{J}) \cap \downarrow K$. □

3 Selector generated models

In [14–16], a framework for defining declarative semantics of generalized logic programs was introduced, which encompasses several other semantics, as already mentioned in the introduction. Parametrization within this framework is done via so-called *selector functions*, defined as follows.

Definition 2. *A* selector *is a function* $\mathrm{Sel} : \mathbf{C}_P \times \mathbf{I}_P \to 2^{\mathbf{I}_P}$*, satisfying* $\emptyset \neq \mathrm{Sel}(C, I) \subseteq [I, \sup(C)]$ *for all* P*-chains* C *and each interpretation* $I \in {\downarrow} \sup(C)$*.*

We use selectors Sel to define nondeterministic successor functions Ω_P on $\mathbf{I}_P$, as follows.

Definition 3. *Given a selector* $\mathrm{Sel} : \mathbf{C}_P \times \mathbf{I}_P \to 2^{\mathbf{I}_P}$ *and a program* P*, the function* Ω_P *is defined by*

$$\Omega_P : (\mathbf{C}_P \times \mathbf{I}_P \to 2^{\mathbf{I}_P}) \times \mathbf{C}_P \times \mathbf{I}_P \to 2^{\mathbf{I}_P}$$
$$\Omega_P(\mathrm{Sel}, C, I) = \mathrm{Min}\left([I, \sup(C)] \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(C, I\right)\right)\right)\right)\right).$$

Example 1. In this paper, we will have a closer look at the following selectors.

lower bound selector	$\mathrm{Sel}_{\mathsf{l}}(C, I) = \{I\}$
lower and upper bound selector	$\mathrm{Sel}_{\mathsf{lu}}(C, I) = \{I, \sup(C)\}$
interval selector	$\mathrm{Sel}_{\mathsf{i}}(C, I) = [I, \sup(C)]$
chain selector	$\mathrm{Sel}_{\mathsf{c}}(C, I) = [I, \sup(C)] \cap C$

With the first two arguments (the selector Sel and the chain C) fixed, the function $\Omega_P(\mathrm{Sel}, C, I)$ can be understood as a nondeterministic consequence operator. Iteration of the function $\Omega_P(\mathrm{Sel}, C, \cdot)$ from the least interpretation $\emptyset$ creates monotonic sequences of interpretations. This leads to the following definition of (P, M, Sel)-chains.

Definition 4. *A* (P, M, Sel)-chain *is a* P*-chain satisfying*

($\mathbf{C}\sup$) $M = \sup(C)$ *and*
($\mathbf{C}\beta_{\mathrm{Sel}}$) $C_{\beta+1} \in \Omega_P(\mathrm{Sel}, C, C_\beta)$ *for all* β*, where* $\beta + 1 < \kappa$ *and* κ *is the length of the transfinite sequence* C*.*

Thus, (P, M, Sel)-chains are monotonic sequences C of interpretations, that reproduce themselves by iterating Ω_P.

Definition 5. *A model* M *of a program* $P \subseteq \mathsf{GLP}$ *is* Sel-generated *if and only if there exists a* (P, M, Sel)*-chain* C*. The* Sel*-semantics of the program* P *is the set* $\mathrm{Mod}_{\mathrm{Sel}}(P)$ *of all* Sel*-generated models of* P*.*

Example 2. Let P be the program consisting of the following rules.

$$\Rightarrow a \qquad (1)$$
$$a \Rightarrow b \qquad (2)$$
$$(a \vee \neg c) \wedge (c \vee \neg a) \Rightarrow c \qquad (3)$$

Then $\{a,b,c\}$ is the only $\mathrm{Sel}_{\mathsf{l}}$-generated model for P, namely via the chain $C_1 = (\emptyset \overset{(1),(3)}{\rightarrow} \{a,c\} \overset{(2)}{\rightarrow} \{a,b,c\}))$. $\{a,b\}$ and $\{a,b,c\}$ are $\mathrm{Sel}_{\mathsf{lu}}$-generated (and $\mathrm{Sel}_{\mathsf{c}}$-generated) models, namlely via the chains $C_2 = (\emptyset \overset{(1)}{\rightarrow} \{a\} \overset{(2)}{\rightarrow} \{a,b\})$ and C_1). $\{a,b\}$ is the only $\mathrm{Sel}_{\mathsf{i}}$-generated model of P, namely via C_2.

Some properties of semantics generated by the selectors in Example 1 were studied in [14]. In Section 5, we will make use of the following results from [14].

Theorem 1. *1. For definite programs $P \subseteq \mathsf{DLP}$, the unique element contained in $\mathrm{Mod}_{\mathsf{l}}(P) = \mathrm{Mod}_{\mathsf{lu}}(P) = \mathrm{Mod}_{\mathsf{c}}(P) = \mathrm{Mod}_{\mathsf{i}}(P)$ is the* least *model of P.*
2. For normal programs $P \subseteq \mathsf{NLP}$, the unique element of $\mathrm{Mod}_{\mathsf{l}}(P)$ is the inflationary *model of P (as introduced in [11]).*
3. For normal programs $P \subseteq \mathsf{NLP}$, the set $\mathrm{Mod}_{\mathsf{lu}}(P) = \mathrm{Mod}_{\mathsf{c}}(P) = \mathrm{Mod}_{\mathsf{i}}(P)$ contains exactly all stable *models of P (as defined in [5]).*
4. For disjunctive programs $P \subseteq \mathsf{DLP}^+$, the minimal elements in $\mathrm{Mod}_{\mathsf{lu}}(P) = \mathrm{Mod}_{\mathsf{c}}(P) = \mathrm{Mod}_{\mathsf{i}}(P)$ are exactly all stable models of P (as defined in [12]), but for generalized programs $P \subseteq \mathsf{GLP}$, the sets $\mathrm{Mod}_{\mathsf{lu}}(P)$, $\mathrm{Mod}_{\mathsf{c}}(P)$, and $\mathrm{Mod}_{\mathsf{i}}(P)$ may differ.
5. For generalized programs $P \subseteq \mathsf{GLP}$, $\mathrm{Mod}_{\mathsf{i}}(P)$ is the set of stable generated models *of P (as defined in [7]).* □

This shows that the framework of selector semantics covers some of the most important declarative semantics for normal logic programs. Selector generated models provide a natural extension of these semantics to generalized logic programs and allow systematic comparisons of many new and well-known semantics.

For all selectors Sel, it was shown in [14] that the Sel-semantics of programs in GLP is invariant with respect to the following transformations. ($\rightarrow_{\mathsf{eq}}$) The replacement of the body and the head of a rule by logically equivalent formulas. ($\rightarrow_{\mathsf{hs}}$) The splitting of conjunctive heads, more precisely the replacement $P \cup \{\varphi \Rightarrow \psi \wedge \psi'\} \ \rightarrow_{\mathsf{hs}} \ P \cup \{\varphi \Rightarrow \psi, \varphi \Rightarrow \psi'\}$. As every formula $\mathsf{head}(r)$ is logically equivalent to a formula in conjunctive normal form, it suffices to study head disjunctive programs.

4 Selector generated models via level mappings

In [8–10], a uniform approach to different semantics for logic programs was given, using the notion of *level mapping*, as follows.

Definition 6. *A* level mapping *for a logic program $P \subseteq \mathsf{GLP}$ is a function $l : \mathsf{B}_P \rightarrow \alpha$, where α is an ordinal.*

In order to display the style of level-mapping characterizations for semantics, we give two examples from [10] which we will further discuss later on.

Theorem 2. *Every definite program $P \subseteq \mathsf{LP}$ has exactly one model M, such that there exists a level mapping $l : \mathsf{B}_P \rightarrow \alpha$ satisfying*

(Fd) *for every atom* $a \in M$ *there exists a rule* $\bigwedge_{b \in B} b \Rightarrow a \in P$ *such that* $B \subseteq M$ *and* $\max\{l(b) \mid b \in B\} < l(a)$.

Furthermore, M *coincides with the least model of* P. □

The following is actually due to Fages [4].

Theorem 3. *Let* P *be a normal program and* M *be an interpretation for* P. *Then* M *is a stable model of* P *iff there exists a level mapping* $l : \mathsf{B}_P \to \alpha$ *satisfying*

(Fs) *for each atom* $a \in M$ *there exists a rule* $r \in P$ *with* $\mathsf{head}(r) = a$, $\mathsf{body}(r)^+ \subseteq M$, $\mathsf{body}(r)^- \cap M = \emptyset$, *and* $\max\{l(b) \mid b \in \mathsf{body}(r)^+\} < l(a)$. □

It is evident, that among the level mappings satisfying the respective conditions in Theorems 2 and 3, there exist pointwise minimal ones.

We set out to prove a general theorem which characterizes selector generated models by means of level mappings, in the style of the results displayed above. The following notion will ease notation considerably.

Definition 7. *For a level mapping* $l : \mathsf{B}_P \to \alpha$ *for a program* $P \subseteq \mathsf{GLP}$ *and an interpretation* $M \subseteq \mathsf{B}_P$, *the (transfinite) sequence* $\mathsf{C}^{l,M}$ *consisting of interpretations of* P *is defined by*

$$\mathsf{C}^{l,M}_\beta = \{a \in M \mid l(a) < \beta\} = M \cap \bigcup_{\gamma < \beta} l^{-1}(\gamma)$$

for all $\beta < \alpha$.

Remark 1. Definition 7 implies the following properties of the (transfinite) sequence $\mathsf{C}^{l,M}$. (1) $\mathsf{C}^{l,M}$ is monotonically increasing, (2) $\mathsf{C}^{l,M}_0 = \emptyset$, and (3) $M = \bigcup_{\beta<\alpha} \mathsf{C}^{l,M}_\beta = \sup \mathsf{C}^{l,M}$.

The following is our main result.

Theorem 4. *Let* $P \subseteq \mathsf{HDLP}$ *be a head disjunctive program and* $M \in \mathbf{I}_P$. *Then* M *is a* Sel*-generated model of* P *iff there exists a level mapping* $l : \mathsf{B}_P \to \alpha$ *satisfying the following properties.*

(L1) $M = \sup\left(\mathsf{C}^{l,M}\right) \in \mathrm{Mod}\,(P)$.

(L2) *For all* β *with* $\beta + 1 < \alpha$ *we have*

$$\mathsf{C}^{l,M}_{\beta+1} \setminus \mathsf{C}^{l,M}_\beta \in \mathrm{Min}\left\{ J \in \mathbf{I}_P \,\middle|\, J \models \mathsf{head}\left(R\left(\mathsf{C}^{l,M}_\beta, J\right)\right)^+ \right\}, \quad \textit{where}$$
$$R\left(\mathsf{C}^{l,M}_\beta, J\right) = \left\{ r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}^{l,M}_\beta\right)\right) \,\middle|\, \begin{array}{l} \mathsf{C}^{l,M}_\beta \not\models \mathsf{head}\,(r)^+ \;\; \textit{and} \\ J \cup \mathsf{C}^{l,M}_\beta \not\models \mathsf{head}\,(r)^- \end{array} \right\}.$$

(L3) *For all limit ordinals* $\lambda < \alpha$ *we have* $\mathsf{C}^{l,M}_\lambda = \bigcup_{\beta<\lambda} \mathsf{C}^{l,M}_\beta$.

Remark 2. As P is a head disjunctive program, we have $\mathsf{C}_\beta^{l,M} \not\models \mathsf{head}(r)^+$ iff $\mathsf{head}(r)^+ \cap \mathsf{C}_\beta^{l,M} = \emptyset$, and $J \cup \mathsf{C}_\beta^{l,M} \not\models \mathsf{head}(r)^-$ iff $\mathsf{head}(r)^- \subseteq J \cup \mathsf{C}_\beta^{l,M}$, thus

$$R\left(\mathsf{C}_\beta^{l,M}, J\right) = \left\{ r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \,\middle|\, \begin{array}{l} \mathsf{head}(r)^+ \cap \mathsf{C}_\beta^{l,M} = \emptyset \text{ and} \\ \mathsf{head}(r)^- \subseteq J \cup \mathsf{C}_\beta^{l,M} \end{array} \right\}.$$

Also note that for every rule $r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \setminus R\left(\mathsf{C}_\beta^{l,M}, J\right)$, we have $\downarrow\left(\mathsf{C}_\beta^{l,M} \cup J\right) \subseteq \mathrm{Mod}\left(\mathsf{head}(r)^-\right)$ or $\uparrow \mathsf{C}_\beta^{l,M} \subseteq \mathrm{Mod}\left(\mathsf{head}(r)^+\right)$. Thus all of these rules are satisfied in the interval $[\mathsf{C}_\beta^{l,M}, \mathsf{C}_\beta^{l,M} \cup J]$.

Proof. (of Theorem 4)
($\Longrightarrow$) By Definition 5, an interpretation M is a Sel-generated model of P iff there exists a (P, M, Sel)-chain. Let M be a model of P and M be Sel-generated by the (P, M, Sel)-chain C of length α.

Define the level mapping $l : \mathsf{B}_P \to \alpha$ by $l(a) = \min\{\beta \mid a \in C_\beta\} - 1$ for all $a \in \mathsf{B}_P$. We show that this function l satisfies **(L1)**,**(L2)** and **(L3)**.

(1) We first show $\mathsf{C}^{l,M} = C$ for the sequence $\mathsf{C}^{l,M}$ determined by l and M according to Definition 7. From Remark 1, we know $\mathsf{C}_0^{l,M} = \emptyset$ and $\sup\left(\mathsf{C}^{l,M}\right) = M$. Moreover, for each $\beta < \alpha$, we have

$$\begin{aligned} \mathsf{C}_\beta^{l,M} &= \{a \in M \mid l(a) < \beta\} \\ &= \{a \in M \mid \min\{\gamma \mid a \in C_\gamma\} - 1 < \beta\} \quad \text{(by Definition 7)} \\ &= \{a \in M \mid a \in C_\gamma, \gamma - 1 < \beta\} \qquad \text{(by definition of } l\text{)} \\ &= C_\beta. \end{aligned}$$

Therefore, we have $\mathsf{C}_\lambda^{l,M} = \bigcup_{\beta<\lambda} \mathsf{C}_\beta^{l,M} = \bigcup_{\beta<\lambda} C_\beta = C_\lambda$ for all limit ordinals $\lambda < \alpha$. This proves $C = \mathsf{C}^{l,M}$.

(2) C is a (P, M, Sel)-chain, so it satisfies **(L1)** and **(L3)**. It remains to show that C satisfies **(L2)**. For all β with $\beta + 1 < \alpha$, we show

(a) $C_{\beta+1} \setminus C_\beta \models \mathsf{head}(R(C_\beta, C_{\beta+1} \setminus C_\beta))^+$ for

$$\begin{aligned} &R(C_\beta, C_{\beta+1} \setminus C_\beta) \\ &= \left\{ r \in \mathsf{fire}(P, \mathrm{Sel}(C, C_\beta)) \,\middle|\, \begin{array}{l} C_\beta \not\models \mathsf{head}(r)^+ \text{ and} \\ C_\beta \cup C_{\beta+1} \setminus C_\beta \not\models \mathsf{head}(r)^- \end{array} \right\} \\ &= \left\{ r \in \mathsf{fire}(P, \mathrm{Sel}(C, C_\beta)) \mid C_\beta \cap \mathsf{head}(r)^+ = \emptyset \text{ and } \mathsf{head}(r)^- \subseteq C_{\beta+1} \right\} \end{aligned}$$

and

(b) for all interpretations $J \subseteq C_{\beta+1} \setminus C_\beta$ where $J \models \mathsf{head}(R(C_\beta, J))^+$, we have $J = C_{\beta+1} \setminus C_\beta$.

(a) C is a (P, M, Sel)-chain, hence we have

$$C_{\beta+1} \in \Omega_P(\mathrm{Sel}, C, C_\beta) = \mathrm{Min}\left([C_\beta, M] \cap \mathrm{Mod}(\mathsf{head}(\mathsf{fire}(P, \mathrm{Sel}(C, C_\beta))))\right),$$

and we obtain $C_{\beta+1} \models \mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)\right)$.

For each rule $r \in R\left(C_\beta, C_{\beta+1} \setminus C_\beta\right)$, we know

$$R\left(C_\beta, C_{\beta+1} \setminus C_\beta\right) \subseteq \mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)$$

and hence $C_{\beta+1} \models \mathsf{head}\left(r\right)$.

By the definition of this set and Remark 2, the set $R\left(C_\beta, C_{\beta+1} \setminus C_\beta\right)$ does not contain any rule $r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)$, where $C_{\beta+1} \models \mathsf{head}\left(r\right)$ is satisfied by $C_{\beta+1} \models \mathsf{head}\left(r\right)^-$ or $C_\beta \models \mathsf{head}\left(r\right)^+$, i.e. $\mathsf{head}\left(r\right)^+ \cap C_\beta \neq \emptyset$. Hence all rules r from $R\left(C_\beta, C_{\beta+1} \setminus C_\beta\right) \subseteq \mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)$ satisfy $C_\beta \models \mathsf{head}\left(r\right)$ by $C_{\beta+1} \setminus C_\beta \cap \mathsf{head}\left(r\right)^+ \neq \emptyset$, i.e. $C_{\beta+1} \setminus C_\beta \models \mathsf{head}\left(r\right)^+$. This shows (a).

(b) Assume $J \subseteq C_{\beta+1} \setminus C_\beta$ and $J \models \mathsf{head}\left(R\left(C_\beta, J\right)\right)^+$. We show $J \cup C_\beta \supseteq C_{\beta+1}$ which implies $J \supseteq C_{\beta+1} \setminus C_\beta$.

First note that $J \cup C_\beta \subseteq \left[C_\beta, M\right] \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)\right)\right)$. Indeed $J \cup C_\beta \in \uparrow C_\beta$ is obvious and $J \cup C_\beta \in \downarrow M$ is implied by $J \subseteq C_{\beta+1} \setminus C_\beta$, i.e. $J \cup C_\beta \subseteq C_{\beta+1}$, and $C_{\beta+1} \subseteq M$ by monotonicity of the chain C.

Now we show $J \cup C_\beta \models \mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)\right)$. Note first that all rules r in the set $\mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)$ satisfy one of the following conditions.

1. $C_\beta \models \mathsf{head}\left(r\right)^+$ (i.e. $C_\beta \cap \mathsf{head}\left(r\right)^+ \neq \emptyset$) and therefore $J \cup C_\beta \models \mathsf{head}\left(r\right)$ by $C_\beta \subseteq J \cup C_\beta$ or
2. $J \cup C_\beta \models \mathsf{head}\left(r\right)^-$ and therefore $J \cup C_\beta \models \mathsf{head}\left(r\right)$ or
3. none of 1. or 2. Then we have $r \in R\left(C_\beta, J\right)$ and due to the assumption $J \in \mathrm{Mod}\left(\mathsf{head}\left(R\left(C_\beta, J\right)\right)^+\right)$ we have $J \cup C_\beta \models \mathsf{head}\left(r\right)^+$ and therefore $J \cup C_\beta \models \mathsf{head}\left(r\right)$.

We can now conclude $J \cup C_\beta \supseteq C_{\beta+1}$ because $C_{\beta+1}$ is a minimal element of $\left[C_\beta, M\right] \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(C, C_\beta\right)\right)\right)\right)$, which proves (b).

Together, we have shown that $C_{\beta+1} \setminus C_\beta$ is a minimal element in

$$\left\{J \in \mathbf{I}_P \mid J \models \mathsf{head}\left(R\left(C_\beta, J\right)\right)^+\right\},$$

which shows that the level mapping l satisfies **(L2)**. This finishes the first part of the proof.

($\Longleftarrow$) For the converse, we show that for every level mapping l for a program P and an interpretation M satisfying **(L1)**,**(L2)** and **(L3)** the sequence $\mathsf{C}^{l,M}$ is a (P, M, Sel)-chain.

Let $l : \mathsf{B}_P \to \alpha$ be a level mapping and M an interpretation for a program P. According to Definition 4, we have to show the following properties of the sequence $\mathsf{C}^{l,M}$:

(C0) $\mathsf{C}_0^{l,M} = \emptyset$,
(Cλ) $\mathsf{C}_\lambda^{l,M} = \bigcup\left\{\mathsf{C}_\beta^{l,M} \mid \beta < \lambda\right\}$ for all limit ordinals $\lambda < \alpha$,
(C sup) $M = \bigcup\left\{\mathsf{C}_\beta^{l,M} \mid \beta < \alpha\right\} = \sup \mathsf{C}^{l,M}$ and
(Cβ_{Sel}) $\mathsf{C}_{\beta+1}^{l,M} \in \Omega_P\left(\mathrm{Sel}, \mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)$ for all β with $\beta + 1 < \alpha$.

By Remark 1 we know that $\mathsf{C}^{l,M}$ increases monotonically and $\mathsf{C}_0^{l,M} = \emptyset$, i.e. $(\mathbf{C0})$, is satisfied. By condition **(L1)** we have $M = \sup\left(\mathsf{C}^{l,M}\right) \in \mathrm{Mod}\,(P)$, i.e. $(\mathbf{C}\sup)$, and condition **(L3)** implies $(\mathbf{C}\lambda)$.

For $(\mathbf{C}\beta_{\mathrm{Sel}})$ we have to show that for all β with $\beta + 1 < \alpha$ the equation

$$\begin{aligned}\mathsf{C}_{\beta+1}^{l,M} &\in \Omega_P\left(\mathrm{Sel}, \mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\\ &= \mathrm{Min}\left(\left[\mathsf{C}_\beta^{l,M}, M\right] \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)\right)\end{aligned}$$

holds. For this, by Lemma 1, it suffices to show that

$$\mathsf{C}_{\beta+1}^{l,M} \in \mathrm{Min}\left(\uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)\right) \cap \downarrow M.$$

Now by monotonicity of $\mathsf{C}^{l,M}$ we know $\mathsf{C}_{\beta+1}^{l,M} \in \downarrow M$. For

$$\mathsf{C}_{\beta+1}^{l,M} \in \mathrm{Min}\left(\uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)\right)$$

we will proceed by proving the steps (a) and (b), as follows.

(a) $\mathsf{C}_{\beta+1}^{l,M} \in \uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)$.

(b) For all interpretations $J \in \uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)$, where $J \subseteq \mathsf{C}_{\beta+1}^{l,M}$, we have $J = \mathsf{C}_{\beta+1}^{l,M}$.

(a) By monotonicity of $\mathsf{C}^{l,M}$ it suffices to show that

$$\mathsf{C}_{\beta+1}^{l,M} \in \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right).$$

First note that for every rule $r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)$ one of the following holds:

1. $\mathsf{C}_\beta^{l,M} \models \mathsf{head}\,(r)^+$, i.e. $\mathsf{head}\,(r)^+ \cap \mathsf{C}_\beta^{l,M} \neq \emptyset$, and therefore $\mathsf{C}_{\beta+1}^{l,M} \models \mathsf{head}\,(r)$ by $\mathsf{C}_\beta^{l,M} \subseteq \mathsf{C}_{\beta+1}^{l,M}$ or
2. $\mathsf{C}_{\beta+1}^{l,M} \models \mathsf{head}\,(r)^-$ and therefore $\mathsf{C}_{\beta+1}^{l,M} \models \mathsf{head}\,(r)$ or
3. none of 1. or 2. Then $r \in R\left(\mathsf{C}_\beta^{l,M}, \mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}\right)$ and thus $\mathsf{C}_{\beta+1}^{l,M} \models \mathsf{head}\,(r)$ by $\mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M} \models \mathsf{head}\,(r)^+$ and condition **(L2)**.

Hence for each rule $r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)$ we have $\mathsf{C}_{\beta+1}^{l,M} \models \mathsf{head}\,(r)$ and thus $\mathsf{C}_{\beta+1}^{l,M} \in \uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)$, which shows (a).

(b) Let $J \in \uparrow \mathsf{C}_\beta^{l,M} \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)$ for some $J \subseteq \mathsf{C}_{\beta+1}^{l,M}$. Since $J \in \uparrow \mathsf{C}_\beta^{l,M}$ we obtain $J \in \uparrow \mathsf{C}_{\beta+1}^{l,M}$ by showing $J \setminus \mathsf{C}_\beta^{l,M} \supseteq \mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}$. Indeed $J \setminus \mathsf{C}_\beta^{l,M} \in \left\{K \in \mathbf{I}_P \mid K \models \mathsf{head}\left(R\left(\mathsf{C}_\beta^{l,M}, K\right)\right)^+\right\}$ and therefore $J \setminus$

$\mathsf{C}_\beta^{l,M} \models \mathsf{head}\left(R\left(\mathsf{C}_\beta^{l,M}, J \setminus \mathsf{C}_\beta^{l,M}\right)\right)^+$. Condition **(L2)**, i.e. minimality of $C_{\beta+1} \setminus \mathsf{C}_\beta^{l,M}$ in this set, implies $J \setminus \mathsf{C}_\beta^{l,M} \supseteq \mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}$ as desired.

By $J \in \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right)$ we have $J \models \mathsf{head}(r)$ for all rules $r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)$. For each of these rules r, $J \models \mathsf{head}(r)$ is satisfied by $J \models \mathsf{head}(r)^-$ or by $\mathsf{C}_\beta^{l,M} \models \mathsf{head}(r)^+$ and in both cases we have $r \notin R\left(\mathsf{C}_\beta^{l,M}, J\right)$. For all remaining rules, we know that $J \models \mathsf{head}(r)$ is satisfied by $J \setminus \mathsf{C}_\beta^{l,M} \cap \mathsf{head}(r)^+ \neq \emptyset$, i.e. $J \setminus \mathsf{C}_\beta^{l,M} \models \mathsf{head}(r)^+$, and therefore we know $J \setminus \mathsf{C}_\beta^{l,M} \in \left\{K \in \mathbf{I}_P \mid K \models \mathsf{head}\left(R\left(\mathsf{C}_\beta^{l,M}, K\right)\right)^+\right\}$.

By $J \setminus \mathsf{C}_\beta^{l,M} \subseteq \mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}$ and minimality of $\mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}$ in the set

$$\left\{K \in \mathbf{I}_P \mid K \models \mathsf{head}\left(R\left(\mathsf{C}_\beta^{l,M}, K\right)\right)^+\right\}$$

we have $J \setminus \mathsf{C}_\beta^{l,M} = \mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M}$ and therefore $J = \mathsf{C}_{\beta+1}^{l,M}$, which shows (b).

This proves the minimality of $\mathsf{C}_{\beta+1}^{l,M}$ in the set

$$[\mathsf{C}_\beta^{l,M}, M] \cap \mathrm{Mod}\left(\mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right)\right).$$

Thus, $\mathsf{C}_{\beta+1}^{l,M} \in \Omega_P\left(\mathrm{Sel}, \mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)$.

Hence $\mathsf{C}^{l,M}$ is a (P, M, Sel)-chain. This proves $M \in \mathrm{Mod}_{\mathrm{Sel}}(P)$ and concludes the proof. □

By the remarks made at the end of Section 3, we obtain the following immediate corollary.

Corollary 1. *Let P be a generalized program and M an interpretation of P. Then M is a* Sel*-generated model of P iff for a head disjunctive program Q with $P \to^*_{\mathsf{eq,hs}} Q$ there exists a level mapping $l : \mathsf{B}_Q \to \alpha$ satisfying **(L1)**, **(L2)** and **(L3)** of Theorem 4.* □

5 Corollaries

We can now apply Theorem 4 in order to obtain level mapping characterizations for every semantics generated by a selector, in particular for those semantics generated by the selectors defined in Example 1 and listed in Theorem 1. For syntactically restricted programs, we can furthermore simplify the properties **(L1)**,**(L2)** and **(L3)** in Theorem 4. Alternative level mapping characterizations for some of these semantics were already obtained directly in [10].

Programs with positive disjunctions in all heads

For rules $r \in \mathsf{HDLP}$, where $\mathsf{head}(r)$ is a disjunction of atoms, we have $\mathsf{head}(r)^- = \emptyset$. Hence we have $\mathsf{head}(r)^- \subseteq I$, i.e. $I \not\models \mathsf{head}(r)^-$, for all interpretations $I \in \mathbf{I}_P$. Thus the set $R\left(\mathsf{C}_\beta^{l,M}, J\right)$ from **(L2)** in Theorem 4 can be specified by

$$R\left(\mathsf{C}_\beta^{l,M}, J\right) = \left\{r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \mid \mathsf{C}_\beta^{l,M} \not\models \mathsf{head}(r)^+\right\}.$$

We furthermore observe that the set $R\left(\mathsf{C}_\beta^{l,M}, J\right)$ does not depend on the interpretation J, so we obtain

$$R'\left(\mathsf{C}_\beta^{l,M}\right) = \left\{r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \mid \mathsf{C}_\beta^{l,M} \cap \mathsf{head}(r)^+ = \emptyset\right\}$$

and hence

$$\mathrm{Min}\left\{J \in \mathbf{I}_P \,\middle|\, J \models \mathsf{head}\left(R\left(\mathsf{C}_\beta^{l,M}, J\right)\right)^+\right\} = \mathrm{Min}\left(\mathrm{Mod}\left(\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\right)\right).$$

Thus for programs containing only rules whose heads are disjunctions of atoms we can rewrite condition **(L2)** in Theorem 4, as follows.

(L2d) For every β with $\beta + 1 < \alpha$ we have

$$\mathsf{C}_{\beta+1}^{l,M} \setminus \mathsf{C}_\beta^{l,M} \in \mathrm{Min}\left(\mathrm{Mod}\left(\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\right)\right), \text{where}$$
$$R'\left(\mathsf{C}_\beta^{l,M}\right) = \left\{r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \,\middle|\, \mathsf{C}_\beta^{l,M} \cap \mathsf{head}(r)^+ = \emptyset\right\}.$$

Programs with atomic heads

Single atoms are a specific kind of disjunctions of atoms. Hence for programs with atomic heads we can replace condition **(L2)** in Theorem 4 by **(L2d)**, and further simplify it as follows.

For rules with atomic heads we have $\mathsf{head}\left(\{r \in P \mid \mathsf{head}(r) \notin I\}\right) = \mathsf{head}(P) \setminus I$ and therefore

$$\begin{aligned}
&\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\\
&= \mathsf{head}\left(\left\{r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \mid \mathsf{head}(r) \cap \mathsf{C}_\beta^{l,M} = \emptyset\right\}\right)\\
&= \mathsf{head}\left(\left\{r \in \mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right) \mid \mathsf{head}(r) \notin \mathsf{C}_\beta^{l,M}\right\}\right)\\
&= \mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}_\beta^{l,M}\right)\right)\right) \setminus \mathsf{C}_\beta^{l,M}.
\end{aligned}$$

Because all formulae in $\mathsf{head}(P)$ are atoms we obtain

$$\begin{aligned}
\mathrm{Min}\left(\mathrm{Mod}\left(\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\right)\right) &= \mathrm{Min}\left(\uparrow\left(\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\right)\right)\\
&= \left\{\mathsf{head}\left(R'\left(\mathsf{C}_\beta^{l,M}\right)\right)\right\}
\end{aligned}$$

and this allows us to simplify **(L2)** in Theorem 4 to the following.

(L2a) For each β with $\beta + 1 < \alpha$ we have

$$\mathsf{C}^{l,M}_{\beta+1} \setminus \mathsf{C}^{l,M}_{\beta} = \mathsf{head}\left(\mathsf{fire}\left(P, \mathrm{Sel}\left(\mathsf{C}^{l,M}, \mathsf{C}^{l,M}_{\beta}\right)\right)\right) \setminus \mathsf{C}^{l,M}_{\beta}.$$

Inflationary models From Section 3 we know that for normal programs P the selector $\mathrm{Sel}_{\mathsf{I}}$ generates exactly the inflationary model of P as defined in [11]. The generalizations of the definition of inflationary models and this result to head atomic programs are immediate. From [16] we also know that every $\mathrm{Sel}_{\mathsf{I}}$-generated model is generated by a $(P, M, \mathrm{Sel}_{\mathsf{I}})$-chain of length ω. Thus level mappings $l : \mathsf{B}_P \to \omega$ are sufficient to characterize inflationary models of head atomic programs. In this case, condition **(L3)** applies only to the limit ordinal $0 < \omega$. But by remark 1, all level mappings satisfy this property. Therefore we do not need condition **(L3)** in the characterization of inflationary models.

Using Theorem 4 and the considerations above, we obtain the following characterization of inflationary models.

Corollary 2. *Let $P \subseteq \mathsf{HDLP}$ be a head atomic program and M an interpretation for P. Then M is the inflationary model of P iff there exists a level mapping $l : \mathsf{B}_P \to \omega$ with the following properties.*

(L1) $M = \sup\left(\mathsf{C}^{l,M}\right) \in \mathrm{Mod}(P)$.
(L2i)

$$\mathsf{C}^{l,M}_{n+1} \setminus \mathsf{C}^{l,M}_{n} = \mathsf{head}\left(\mathsf{fire}\left(P, \mathsf{C}^{l,M}_{n}\right)\right) \setminus \mathsf{C}^{l,M}_{n}$$

for all $n < \omega$. □

Normal programs

For normal programs, the heads of all rules are single atoms. Hence the simplification **(L2a)** of condition **(L2)** in Theorem 4 applies for all selector generated semantics for normal programs.

The special structure of the bodies of all rules in normal programs allows an alternative formulation of **(L2a)**. In every normal rule, the body is a conjunction of literals. Thus for any set of interpretations $\mathbf{J}$ we have $\mathbf{J} \models \mathsf{body}(r)$ iff $\mathsf{body}(r)^+ \subseteq J$ and $\mathsf{body}(r)^- \cap J = \emptyset$ for all interpretations $J \in \mathbf{J}$.

Stable models We develop next a characterization for stable models of normal programs, as introduced in [5]. The selector $\mathrm{Sel}_{\mathsf{lu}}$ generates exactly all stable models for normal programs. In [16], it was also shown that all $\mathrm{Sel}_{\mathsf{lu}}$-generated models M of a program P are generated by a (P, M, Sel)-chain of length $\leq \omega$. So for the same reasons as discussed for inflationary models, level mappings with range ω are sufficient to characterize stable models and condition **(L3)** can be neglected.

For a normal rule r and two interpretations $I, M \in \mathbf{I}_P$ with $I \leq M$ we have $\{I, M\} \models \mathsf{body}(r)$, i.e. $I \models \mathsf{body}(r)$ and $M \models \mathsf{body}(r)$, iff $\mathsf{body}(r)^+ \subseteq I$ and $\mathsf{body}(r)^- \cap M = \emptyset$. Combining this with **(L2a)** we obtain the following characterization of stable models for normal programs.

Corollary 3. *Let $P \subseteq \mathsf{NLP}$ be a normal program and M an interpretation for P. Then M is a stable model of P iff there exists a level mapping $l : \mathsf{B}_P \to \omega$ satisfying the following properties:*

(L1) $M = \sup\left(\mathsf{C}^{l,M}\right) \in \mathrm{Mod}(P)$.
(L2s)

$$\mathsf{C}^{l,M}_{n+1} \setminus \mathsf{C}^{l,M}_{n} = \mathsf{head}\left(\left\{r \in P \mid \mathsf{body}(r)^+ \subseteq \mathsf{C}^{l,M}_{n}, \mathsf{body}(r)^- \cap M = \emptyset\right\}\right) \setminus \mathsf{C}^{l,M}_{n}$$

for all $n < \omega$. □

Comparing this with Theorem 3, we note that both theorems characterize the same set of models. Thus for a model M of P there exists a level mapping $l : \mathsf{B}_P \to \alpha$ satisfying **(L1)** and **(L2s)** iff there exists a level mapping $l : \mathsf{B}_P \to \omega$ satisfying **(Fs)**. The condition imposed on the level mapping in Theorem 3, however, is weaker than the condition in Corollary 3, because levelmappings defined by (P, M, Sel)-chains are always pointwise minimal.

Definite programs

In order to characterize the least model of definite programs, we can further simplify condition **(L2)** in Theorem 4. Definite programs are a particular kind of head atomic programs. Thus we can replace condition **(L2)** in Theorem 4 by **(L2i)**. Since the body of every definite rule is a conjunction of atoms we obtain

$$\mathsf{fire}(P, I) = \left\{r \in P \mid \mathsf{body}(r)^+ \subseteq I\right\}$$

for every interpretation $I \in \mathbf{I}_P$. Thus we get the following result.

Corollary 4. *Let $P \subseteq \mathsf{LP}$ be a definite program and M an interpretation for P. Then M is the least model of P iff there exists a level mapping $l : \mathsf{B}_P \to \omega$ satisfying the following conditions.*

(L1) $M = \sup\left(\mathsf{C}^{l,M}\right) \in \mathrm{Mod}(P)$.
(L2l)

$$\mathsf{C}^{l,M}_{n+1} \setminus \mathsf{C}^{l,M}_{n} = \mathsf{head}\left(\left\{r \in P \mid \mathsf{body}(r)^+ \subseteq \mathsf{C}^{l,M}_{n}\right\}\right) \setminus \mathsf{C}^{l,M}_{n}$$

for every $n < \omega$. □

Comparing this to Theorem 2, we note that the relation between the conditions **(L2l)** and **(Fd)** are similar to those of the conditions **(Fs)** und **(L2s)**.

6 Conclusions and Further Work

Our main result, Theorem 4, provides a characterization of selector generated models — in general form — by means of level mappings in accordance with the uniform approach proposed in [8–10]. As corollaries from this theorem, we have also achieved level mapping characterizations of several semantics encompassed by the selector generated approach due to [14–16].

Our contribution is technical, and provides a first step towards a comprehensive comparative study of different semantics of logic programs under extended syntax by means of level mapping characterizations. Indeed, a very large number of syntactic extensions for logic programs are currently being investigated in the community, and even for some of the less fancy proposals there is often no agreement on the preferable way of assigning semantics to these constructs.

A particularly interesting case in point is provided by disjunctive and extended disjunctive programs, as studied in [6]. While there is more or less general agreement on an appropriate notion of stable model, as given by the notion of *answer set* in [6], there exist various different proposals for a corresponding well-founded semantics, see e.g. [17]. We expect that recasting them by means of level-mappings will provide a clearer picture on the specific ways of modelling knowledge underlying these semantics.

Eventually, we expect that the study of level mapping characterizations of different semantics will lead to methods for extracting other, e.g. procedural, semantic properties from the characterizations, like complexity or decidability results.

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