# Computational Complexity

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## Contents

# 1 Big-Oh-Notation

[4, Chapter 14.2]

$\mathbb{N} = \{0, 1, 2, \dots\}$
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{R}$: the real numbers

**1.1 Definition**
$f, g : \mathbb{N} \to \mathbb{N}$
$f$ is *of order* $g$, if there is a constant $c > 0$ and $n_0 \in \mathbb{N}$ s.t. $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
$O(g) := \{f \mid f \text{ is of order } g\}$ ("big oh of $g$").
If $f \in O(g)$ we can say that *$g$ provides an asymptotic upper bound on $f$*.
If $f \in O(g)$ and $g \in O(g)$, then they *have the same rate of growth*, and *$g$ is an asymptotically tight bound on $f$* (and vice versa).

**1.2 Remark**
Common abuse of notation:
$f = O(g)$ instead of $f \in O(g)$.
$f(n) = n^2 + O(n)$ instead of "$f(n) = n^2 + g(n)$ for some $g \in O(n)$".

**1.3 Example**
$f(n) = n^2$; $g(n) = n^3$
$f \in O(g)$. [For $c = 1$ and $n > 1$, $n^2 \leq c \cdot n^3$.]
$g \notin O(f)$. 03/31/10
[Assume $n^3 \in O(n^2)$. Then ex. $c, n_0$ s.t. $n^3 \leq c \cdot n^2$ for all $n \geq n_0$. Choose $n_1 = 1 + \max\{c, n_0\}$. Then $n_1^3 = n_1 \cdot n_1^2 > c \cdot n_1^2$ and $n_1 > n_0$. ↯ ]

**Exercise 1 (Due 04/05/10)** Show that $2^n \in O(n!)$.

**Exercise 2** Show that $n! \notin O(2^n)$.

**Exercise 3** Show: If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

**1.4 Example**
$f(n) = n^2 + 2n + 5$; $g(n) = n^2$
$g \in O(f)$ [For $c = 1$ and $n > 0$, $n^2 \leq c \cdot (n^2 + 2n + 5)$.]
$f \in O(g)$
[For $n > 1$ we have $f(n) = n^2 + 2n + 5 \leq n^2 + 2n^2 + 5n^2 = 8n^2 = 8 \cdot g(n)$. Hence, for $c = 8$ and $n > 1$, $f(n) \leq c \cdot g(n)$.]

**1.5 Definition**
$\Theta(g) := \{f \mid f \in O(g) \text{ and } g \in O(f)\}$

**1.6 Example**
For $f, g$ from Example 1.4, $f \in \Theta(g)$.

**1.7 Remark**
A *polynomial* (with integer coefficients) *of degree* $r \in \mathbb{N}$ is a function of the form

$$f : \mathbb{N} \to \mathbb{Z} : n \mapsto c_r \cdot n^r + c_{r-1} \cdot n^{r-1} + \cdots + c_1 \cdot n + c_0,$$

with $0 < r \in \mathbb{N}$, *coefficients* $c_i \in \mathbb{Z}$ $(i = 1, \ldots, r)$, $c_r \neq 0$.
For $f, g : \mathbb{N} \to \mathbb{Z}$, we say $f \in O(g)$ if $|f| \in O(|g|)$, where $|f| : \mathbb{N} \to \mathbb{N} : n \mapsto |f(n)|$.

**1.8 Example**
$f(n) = n^2 + 2n + 5$; $g(n) = -n^2$
$g \in O(f)$
[We have $|g| : n \mapsto n^2$ and $|f| \equiv f$. Thus, from Example 1.4 we know that $|g| \in O(|f|)$.]
$f \in O(g)$ [As before, from Example 1.4.]

**1.9 Remark**
For $f, g : \mathbb{N} \to \mathbb{R}$, we say $f \in O(g)$ if $\lfloor f \rfloor \in O(\lfloor g \rfloor)$.

**1.10 Remark**
$\log_a(n) \in O(\log_b(n))$ for all $1 < a, b \in \mathbb{N}$. [$\log_a(n) = \log_a(b) \cdot \log_b(n)$ for all $n \in \mathbb{N}$.]

**1.11 Theorem**
The following hold.

1. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \in O(g)$ and $g \notin O(f)$.
2. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c$ with $0 < c < \infty$, then $f \in \Theta(g)$ and $g \in \Theta(f)$.
3. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \notin O(g)$ and $g \in O(f)$.

**Proof:** We show part 1.
Assume $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$, i.e., for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ we have $\frac{f(n)}{g(n)} < \varepsilon$, and hence $f(n) < \varepsilon g(n)$. Now select $c = \varepsilon = 1$ and $n_0 = n_\varepsilon$. Then $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$, which shows $f \in O(g)$.
Now if we also assume $g \in O(f)$, then there must exist $d > 0$ and $m_0 \in \mathbb{N}$ s.t. $g(n) \leq d \cdot f(n)$ for all $n \geq m_0$, i.e., $\frac{1}{d} \leq \frac{f(n)}{g(n)}$ for all $n \geq m_0$. But then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} \geq \frac{1}{d} > 0$ ↯ . ∎

**Exercise 4** Show Theorem 1.11 part 2.

**Exercise 5** Show Theorem 1.11 part 3. [Hint: Use part 1.]

**1.12 Remark**
The l'Hospital's Rule often comes in handy:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

**1.13 Example**
$n\log_a(n) \in O(n^2)$ and $n^2 \notin O(n\log_a(n))$

$$\left[\lim_{n\to\infty} \frac{n\log_a(n)}{n^2} = \lim_{n\to\infty} \frac{\log_a(n) + n(\log_a(e)/n)}{2n} = \lim_{n\to\infty} \frac{\log_a(n)}{2n} + \lim_{n\to\infty} \frac{\log_a(e)}{2n} = 0 + 0 = 0\right]$$

**1.14 Theorem**
Let $f$ be a polynomial of degree $r$. Then

(1) $f \in \Theta(n^r)$

(2) $f \in O(n^k)$ for all $k > r$

(3) $f \notin O(n^k)$ for all $k < r$

**Exercise 6** Prove Theorem 1.14 (1). [Hint: Use l'Hospital's Rule.]

**1.15 Theorem**
The following hold.

(1) $\log_a(n) \in O(n)$ and $n \notin O(\log_a(n))$

(2) $n^r \in O(b^n)$ and $b^n \notin O(n^r)$

(3) $b^n \in O(n!)$ and $n! \notin O(b^n)$

**Exercise 7** Prove Theorem 1.15 (1).

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**1.16 Remark**
The Big Oh Hierarchy.

| | | | |
|---|---|---|---|
| $O(1)$ | constant | "sublinear" | "subpolynomial" |
| $O(\log_a(n))$ | logarithmic | "sublinear" | "subpolynomial" |
| $O(n)$ | linear | | "subpolynomial" |
| $O(n\log_a(n))$ | $n\log n$ | | "subpolynomial" |
| $O(n^2)$ | quadratic | | "polynomial" |
| $O(n^3)$ | cubic | | "polynomial" |
| $O(n^r)$ | polynomial ($r \geq 0$) | | |
| $O(b^n)$ | exponential ($b > 1$) | | "exponential" |
| $O(n!)$ | factorial | | "exponential" |

# 2 Turing Machines and Time Complexity

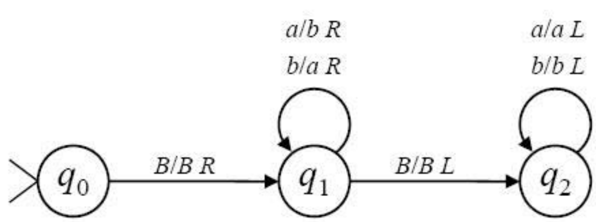
[4, Chapter 14.3 and recap Chapters 8.1, 8.2]

**2.1 Definition**
A standard (*single tape, deterministic*) *Turing machine* (TM) is a quintuple $M = (Q, \Sigma, \Gamma, \delta, q_0)$ with
$Q$ a finite set of *states*
$\Gamma$ a finite set called *tape alphabet* containing a *blank* $B$
$\Sigma \subseteq \Gamma \setminus \{B\}$ the *input alphabet*
$\delta : Q \times \Gamma \overset{\text{partial}}{\to} Q \times \Gamma \times \{L, R\}$, the *transition function*
$q_0 \in Q$ the *start state*

Recall:

- Tape has a left boundary and is infinite to the right.
- Tape positions numbered starting with 0.
- Each tape position contains an element from $\Gamma$.
- Machine starts in state $q_0$ and at position 0.
- Input is written on tape beginning at 1.
- Rest of tape is blank.
- A transition
  1. changes state,
  2. writes new symbol at tape position,
  3. moves head left or right.
- Computation halts if no action is defined.
- Computation terminates abnormally if it moves left of position 0.
- TMs can be represented by state diagrams.

**2.2 Example**
Swap all $a$'s to $b$'s and all $b$'s to $a$'s in a string of $a$'s and $b$'s.

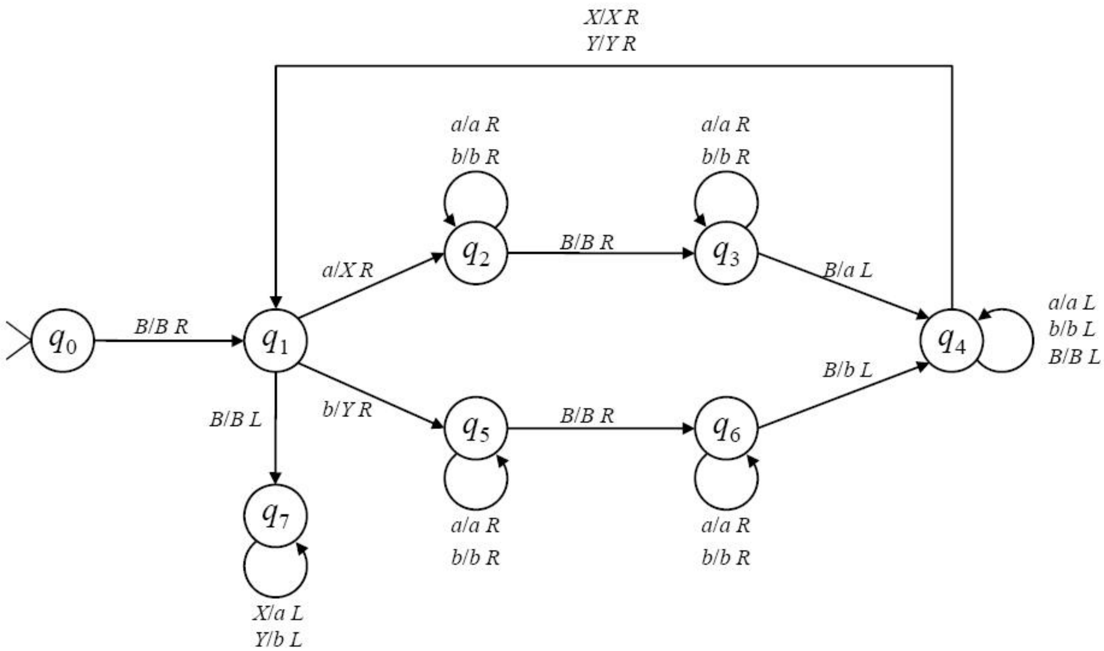



Computation example:

$$\begin{array}{llll} \vdash q_0BababB & \vdash Bbaq_1abB & \vdash Bbabq_2aB & \\ \vdash Bq_1ababB & \vdash Bbabq_1bB & \vdash Bbaq_2baB & \vdash Bq_2babaB \\ \vdash Bbq_1babB & \vdash Bbabaq_1B & \vdash Bbq_2abaB & \vdash q_2BbabaB \end{array}$$

Number of steps required with input string size $n$: $n + 2 + n + 1 = 2n + 3$

**2.3 Example**
Copying a string: $BuB$ becomes $BuBuB$ ($u$ is a string of $a$'s and $b$'s)

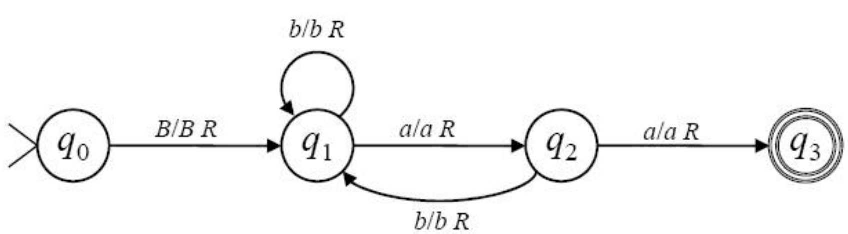



Number of steps required with input string size $n$: $O(n^2)$.

**Exercise 8** Make a standard TM which moves an input string consisting of $a$'s and $b$'s one position to the right. What is the complexity of your TM?

Language accepting TMs additionally have a set $F \subseteq Q$ of *final* states. (They are sextuples.) A string is accepted if the computation halts in a final state (and does not terminate abnormally).

**2.4 Example**
A TM for $(a \cup b)^* aa(a \cup b)^*$.




Number of steps required with input string size $n$: $n$ (worst case)

**2.5 Example**
A TM for $\{a^i b^i c^i \mid i \geq 0\}$ (Figure 1).
Number of steps required with input string size $n = 3i$: $O(i \cdot 4i) = O(i^2) = O(n^2)$ (worst case)

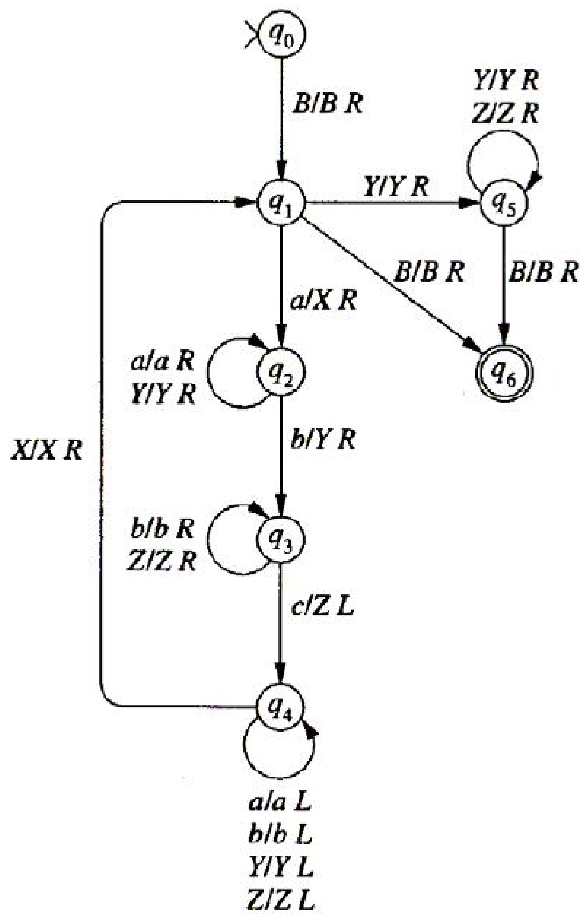



Figure 1: A TM for $\{a^i b^i c^i \mid i \geq 0\}$

**Exercise 9** Make a standard TM which accepts the language $\{a^{2i}b^i \mid i \geq 0\}$. What is the complexity of your TM?

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**2.6 Example**
A TM for palindromes over $a$ and $b$ (Figure 2).
Number of steps required with input string size $n$:

$$\sum_{i=1}^{n+1} i = \frac{1}{2} \cdot (n+2)(n+1) \in O(n^2) \qquad \text{(worst case)}$$

**2.7 Definition**
For any TM $M$, the *time complexity* of $M$ is the function $tc_M : \mathbb{N} \to \mathbb{N}$ s.t. $tc_M(n)$ is the maximum number of transitions processed by a computation of $M$ on input of length $n$.

**2.8 Definition**
A language $L$ is *accepted in deterministic time* $f(n)$ if there is a single tape deterministic TM $M$ for it with $tc_M(n) \in O(f(n))$.

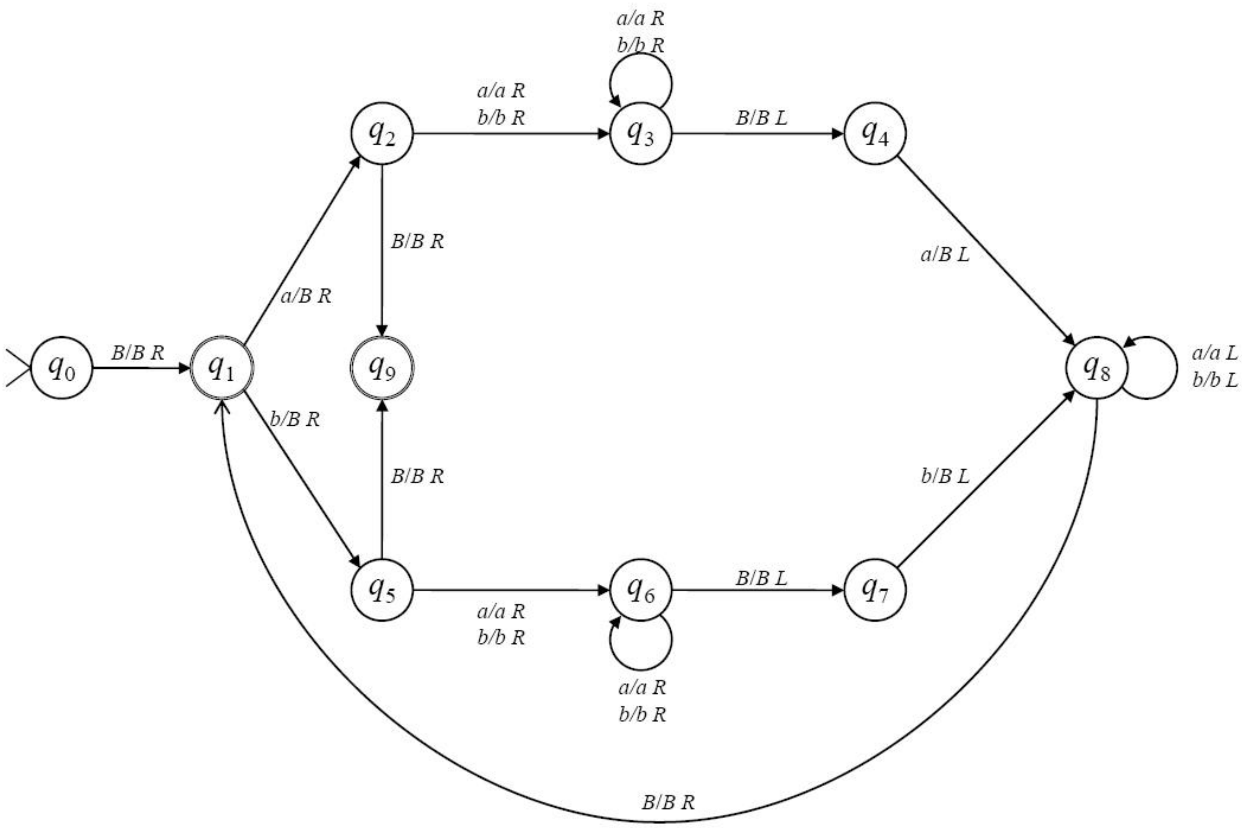



Figure 2: A TM for palindromes over $a$ and $b$.

**Exercise 10 (Due 04/12/10)** Design a single tape TM $M$ for $\{a^i b^i \mid i \geq 0\}$ with $tc_M \in O(n \log_2(n))$. [Hint: On each pass, mark half of the $a$'s and $b$'s that have not been previously marked.]

**2.9 Remark**
Note, that worst-case behavior can happen when a string is *not* accepted. [E.g., straightforward TM to accept strings containing an $a$.]

# 3 Complexity under Turing Machine Variations

[4, Chapter 14.3 cont., 14.4 and recap Chapters 8.5, 8.6]

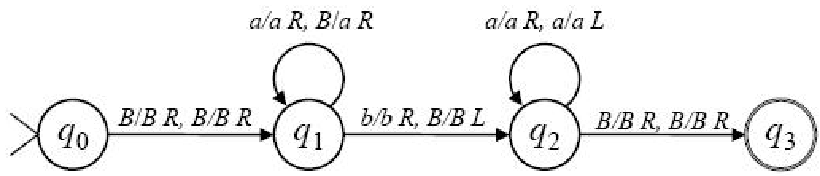
A *k-track TM* has one tape with $k$ tracks. A single read-write head simultaneously reads the $k$ symbols at the head position from all $k$ tracks. We write the transitions as $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}$. The input string is on track 1.

**3.1 Theorem**
If $L$ is accepted by a $k$-track TM $M$ then there is a standard TM $M'$ s.t. $tc_{M'}(n) = tc_M(n)$.

**Proof:** For $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$, let $M' = (Q, \Sigma \times \{B\}^{k-1}, \Gamma^k, \delta', q_0, F)$ with $\delta'(q_i, (x_1, \ldots, x_k)) = \delta(q_i, [x_1, \ldots, x_k])$. The number of transitions needed for a computation is unchanged. ■

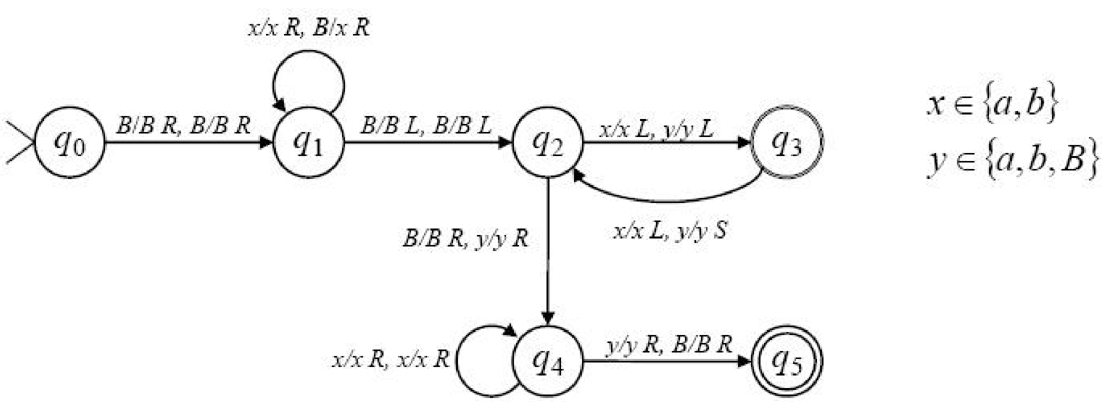
A *k-tape TM* has $k$ tapes and $k$ independent tape heads, which read simultaneously. A transision (i) changes the state, (ii) writes a symbol on each tape, and (iii) independently moves all tape heads. Transitions are written $\delta : (q_i, x_1, \ldots, x_k) \mapsto [q_j; y_1, d_1; \ldots; y_k, d_k]$ where $x_l, y_l \in \Gamma$ and $d_i \in \{L, R, S\}$ ($S$ means head stays). Any head moving off the tape causes an abnormal termination. The input string is on tape 1.

**3.2 Example**
2-tape TM for $\{a^iba^i \mid i \geq 0\}$.



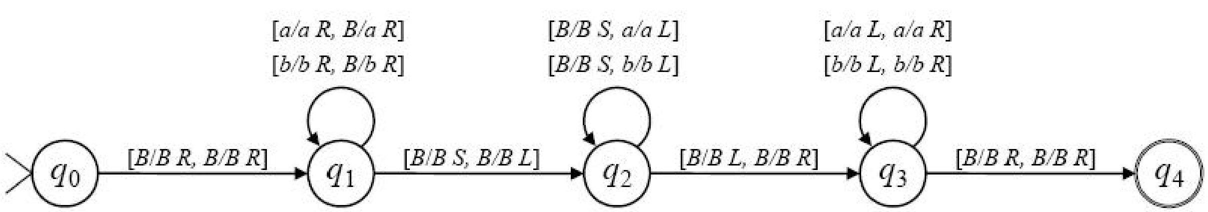

Number of steps required with input string size $n$: $n + 2$

**3.3 Example**
2-tape TM for $\{uu \mid u \in \{a, b\}^\star\}$.




Number of steps required with input string size $n$: $\frac{5}{2}n + 4$

**3.4 Example**
2-tape TM accepting palindromes.



Number of steps required with input string size $n$: $3(n+1)+1$

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**3.5 Theorem**
If $L$ is accepted by a $k$-tape TM $M$ then there is a standard TM $N$ s.t. $tc_N(n) = O(tc_M(n)^2)$.

**Proof:** (sketch) By Theorem 3.1 it suffices to construct an equivalent $2k+1$-track TM $M'$ s.t. $tc_{M'} \in O(tc_M(n)^2)$.

**Simulation of the TM:**

We show how to do this for $k = 2$ (but the argument generalizes).

Idea: tracks 1 and 3 maintain info on tapes 1 and 2 of $M$; tracks 2 and 4 have a single nonblank position indicating the position of the tape heads of $M$.

Initially: write # in track 5, position 1 and $X$ in tracks 2 and 4, position 1.
States: 8-tuples of the form $[s, q_i, x_1, x_2, y_1, y_2, d_1, d_2]$, where $q_i \in Q$, $x_i, y_i \in \Sigma \cup \{U\}$, $d_i \in \{L, R, S, U\}$. $s$ represents the status of the simulation. $U$ indicates an unknown item.

Let $\delta : (q_i, x_1, x_2) \mapsto [q_j; y_1, d_1; y_2, d_2]$ be the applicable transition of $M$.
$M'$ start state: $[f1, q_i, U, U, U, U, U, U]$. The following actions simulate the transition of $M$:

1. $f1$ (find first symbol): $M'$ moves to the right until $X$ on track 2.
   Enter state $[f1, q_i, x_1, U, U, U, U, U]$, where $x_1$ is symbol on track 1 under $x$.
   $M'$ returns to the position with # in track 5.
2. $f2$ (find second symbol): Same as above for recording symbol $x_2$ in track 3 under $X$ in track 4.
   Enter state $[f2, q_i, x_1, x_2, U, U, U, U]$.
   Tape head returns to #.
3. Enter state $[p1, q_j, x_1, x_2, y_1, y_2, d_1, d_2]$, with $q_j, y_1, y_2, d_1, d_2$ obtained from $\delta(q_i, x_1, x_2)$.
4. $p1$ (print first symbol): move to $X$ in track 2.
   Write symbol $y_1$ on track 1. Move $X$ on track 2 in direction indicated by $d_1$.
   Tape head returns to #.
5. $p2$ (print second symbol): move to $X$ in track 4.
   Write symbol $y_2$ on track 3. Move $X$ on track 4 in direction indicated by $d_2$.
   Tape head returns to #.

If $\delta(q_i, x_1, x_2)$ is undefined, then simulation halts after step 2. $[f2, q_i, x_1, y_1, U, U, U, U]$ is accepting whenever $q_i$ is accepting.

For each additional tape, add two trackes, and states obtain 3 more parameters. The simulation has 2 more actions (a find and a print for the new tape).

**Complexity analysis:**

Assume we simulate the $t$-th transition of $M$.
Heads of $M$ are maximally at positions $t$.
Finds require maximum of $k \cdot 2t$ steps.
Prints require maximum of $k \cdot 2(t+1)$ steps.
Simulation of $t$-th transition requires maximum of $4kt + 2k + 1$ steps.

Thus

$$tc_{M'}(n) \leq 1 + \sum_{t=1}^{f(n)} (4kt + 2k + 1) \in O(f(n)^2).$$

■

**Exercise 11** Let $M$ be the TM from Example 3.2 and let $N$ (standard 1-track!) be constructed as in the proof of Theorem 3.5. Determine the number of states of $N$.

**Exercise 12** Let a *Random Access TM* (RATM) be a one-tape TM where transitions are of the form $\delta(q_i, x) = (q_j, y, d)$, where $d \in \mathbb{N}$. Such a transition is performed as usual, but the tape head is then moved to position $d$ on the tape.
Give a sketch, how a RATM can be simulated by a standard TM.

**Exercise 13** Do you think that the following is true?
*For any RATM $M$ there is a standard TM $N$ such that $tc_N(n) = O(tc_M(n))$.*
Justify your answer (no full formal proof needed).

# 4 Linear Speedup

[4, Chapter 14.5]

**4.1 Theorem**
Let $M$ be a $k$-tape TM, $k > 1$, that accepts $L$ with $tc_M(n) = f(n)$. Then, for any $c > 0$, there is a $k$-tape TM $N$ that accepts $L$ with $tc_N(n) \leq \lceil c \cdot f(n) \rceil + 2n + 3$.

**4.2 Corollary**
Let $M$ be a 1-tape TM that accepts $L$ with $tc_M(n) = f(n)$. Then, for any $c > 0$, there is a 2-tape TM $N$ that accepts $L$ with $tc_N(n) \leq \lceil c \cdot f(n) \rceil + 2n + 3$.

**Proof:** The 1-tape TM can be understood as a 2-tape TM where tape 2 is not used. Then apply Theorem 4.1. ■

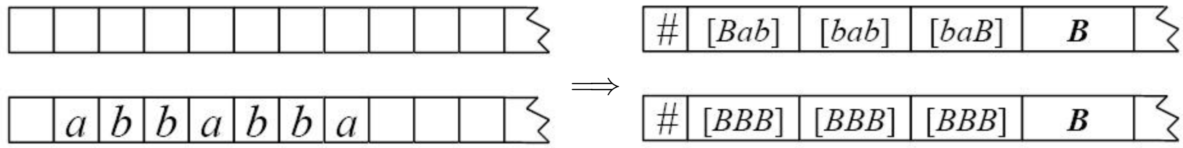
**Exercise 14** Assume a language $L$ is accepted by a 2-tape TM $M$ with $tc_M(n) = \sqrt{n}$. Do you think it is possible to design a standard TM $N$ for $L$ with $tc_N(n) \in O(n)$? Justify your answer.

**Proof sketch/idea for Theorem 4.1**:
Exemplified using Example 3.4.
$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$.

**Simulation of the TM:**

$N$ input alphabet: $\Sigma$.
$N$ tape alphabet: $\Gamma \cup \{\#\} \cup \Gamma^m$

Initialization of $N$ by example.



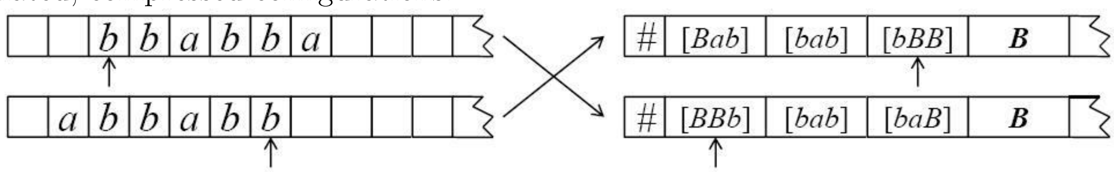

A state of $N$ consists of:

- the state of $M$
- for $i = 1, \ldots, k$, the $m$-tuple currently scanned on tape $i$ of $N$ and the $m$-tuples to the immediate right and left
- a $k$-tuple $[i_1, \ldots, i_k]$, where $i_j$ is the position of the symbol on tape $j$ being scanned by $M$ in the $m$-tuple being scanned by $N$.

After initialization, enter state

$$(q_0; ?, [BBB], ?; ?, [Bab], ?; [1, 1])$$

(? is a placeholder, filled in by subsequent movements).
Idea: Six transitions of $N$ simulate $m$ transitions of $M$ ($m$ depends on $c$).
Simulated, compressed configurations:

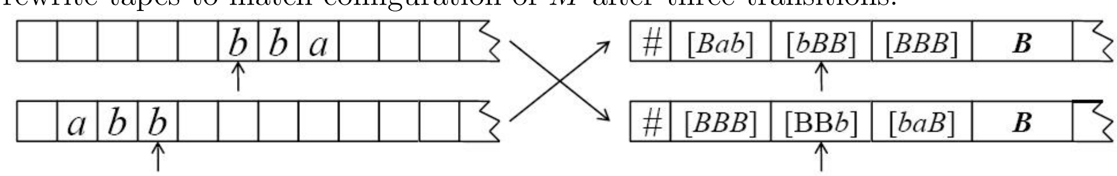



State:

$$(q3; ?, [BBb], ?; ?, [bBB], ?; [3, 1])$$

Six transitions are performed:

1. move left, record tuples in the state
2. move two to the right, record tuples in the state
3. move one to the left
   State: $(q_3, \#, [BBb], [bab]; [bab], [bBB], [BBB]; [3, 1])$
4. and
5. rewrite tapes to match configuration of $M$ after three transitions.




   State: $(q_3, ?, [BBb], ?; ?, [bBB], ?; [3, 1])$

**Complexity analysis:**

Initialization requires $2n + 3$ transitions.
For simulation, 6 moves of $N$ simulate $m$ moves of $M$.
With $m \geq \frac{6}{c}$ we obtain

$$tc_N(n) = \left\lceil \frac{6}{m} f(n) \right\rceil + 2n + 3 \leq \lceil c \cdot f(n) \rceil + 2n + 3$$

as desired.

**4.3 Remark**
In Theorem 4.1, the speedup is obtained by using a larger tape alphabet and by vastly increasing the number of states.

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# 5 Properties of Time Complexity

[4, Chapter 14.6]

**5.1 Theorem**
Let $f$ be a total computable function. Then there is a language $L$ such that $tc_M$ is not bounded by $f$ for any TM that accepts $L$.

**Proof sketch:** Let $u$ be an injective function which assigns to every TM $M$ with $\Sigma = \{0, 1\}$ a string $u(M)$ over $\{0, 1\}$. [Each TM can be described by a finite string of characters; then consider a bit encoding of the characters.]
Let $u_1, u_2, u_3, \ldots$ be an enumeration of all strings over $\{0, 1\}$. If $u(M) = u_i$ for some $M$, then set $M_i = M$, otherwise set $M_i$ to be the one-state TM with no transitions.
This needs to be done, such that there is a TM $N$ which, on input any $u_i$, can simulate $M_i$.

$$L = \{u_i \mid M_i \text{ does not accept } u_i \text{ in } f(\text{length}(u_i)) \text{ or fewer moves}\}.$$

$L$ is recursive. [Input some $u_i$. Determine length($u_i$). Compute $f(\text{length}(u_i))$. Simulate $M_i$ on $u_i$ until $M_i$ either halts or completes $f(\text{length}(u_i))$ transitions, whichever comes first. $u_i$ is accepted if either $M_i$ halted without accepting $u_i$ or $M_i$ did not halt in the first $f(\text{length}(u_i))$ transitions. Otherwise, $u_i$ is rejected.]
Let $M$ be a TM accepting $L$. Then $M = M_j$ for some $j$. $M = M_j$ accepts $u_j$ iff $M_j$ halts on input $u_j$ without accepting $u_j$ in $f(\text{length}(u_j))$ or fewer steps or $M_j$ does not halt in the first $f(\text{length}(u_j))$ steps.
Hence, if $M$ accepts $u_j$ then it needs more than $f(\text{length}(u_j))$ steps.
If $M$ does not accept $u_j$, then it also cannot stop in $f(\text{length}(u_j))$ or fewer steps, since then it would in fact accept $u_j$. ■

04/19/10

**5.2 Theorem**
There is a language $L$ such that, for any TM $M$ that accepts $L$, there is a TM $N$ that accepts $L$ with $tc_N(n) \in O(\log_2(tc_M(n)))$.

**Proof:** skipped ■

**Exercise 15 (Due 04/28/10)** Is the following true or false? Prove your claim.
*For the language $L$ from Theorem 5.2, the following holds: If there is a TM $M$ that accepts $L$ with $tc_M(n) \in O(2^n)$, then there is a TM $N$ that accepts $L$ with $tc_N(n) \in O(n)$.*

**Exercise 16 (Due 04/28/10)** Is the following true or false? Prove your claim.
*Let $M$ be a 5-track TM which accepts a language $L$. Then there is a 5-tape TM $N$ that accepts $L$ with*

$$tc_N(n) \leq \frac{7 + 7n + tc_M(n)}{2}.$$

# 6 Simulation of Computer Computations

[4, Chapter 14.7]

Is the TM complexity model adequate?
Assume a computer with the following parameters.

- finite memory divided into word-size chunks
- fixed word length, $m_w$ bits each
- each word has a fixed numeric address
- finite set of instructions
- each instruction
  - fits in a single word
  - has maximum $t$ operands (addresses used in operation)
  - can do one of
    - move data
    - perform arithmetic or Boolean calculations
    - adjust the program flow
    - allocate additional memory (maximum of $m_a$ words each time)
  - can change at most $t$ words in the memory

Now simulate in $5+t$-tape TM. Tapes are:

- Input tape (divided into word chunks)
- Memory counter (address of next free word on tape 1)
- Program counter (location of next instruction to be exectuted)
- Input counter (location of beginning of input and location of next word to be read)
- Work tape
- $t$ Register tapes

Simulation of $k$-th instruction:

1. load operand data onto the register tapes (max $t$ words)
2. perform indicated operation (one)
3. store results as requried (stores max $t$ words or allocates $m_a$ words or memory)

Operation: finite number of instructions with at most $t$ operands. Maximum number of transitions needed shall be $t_0$ (max exists and is $< \infty$)
Load and Store:
$m_p$ number of bits used to store input
$m_i$ number of bits used to store instructions
$m_k$ total memory allocated by $TM$ before instruction $k$

$$m_k = m_p + m_i + k \cdot m_a$$

Any address can be located in $m_k$ transitions.
Upper bounds:

| | |
|---|---|
| find instruction | $m_k$ |
| load operands | $t \cdot m_k$ |
| return register tape heads | $t \cdot m_k$ |
| perform operation | $t_0$ |
| store information | $t \cdot m_{k+1}$ |
| return register tape heads | $t \cdot m_{k+1}$ |

upper bound for $k$-th instruction:

$$(2t+1)m_k + 2tm_{k+1} + t_0 = (4t+1)m_p + (4t+1)m_i + 2tm_a + t_0 + (4t+1)km_a$$

If computer requires $f(n)$ steps on input length $m_i = n$, then simulation requires:

$$\begin{aligned}
&\sum_{k=1}^{f(n)}((4t+1)m_p + (4t+1)n + 2tm_a + t_0 + (4t+1)km_a) \\
&\qquad = f(n)((4t+1)m_p + (4t+1)n + 2tm_a + t_0) + \sum_{k=1}^{f(n)}(4t+1)km_a \\
&\qquad = f(n)((4t+1)m_p + (4t+1)n + 2tm_a + t_0) + (4t+1)m_a\sum_{k=1}^{f(n)}k \\
&\qquad \in O(\max\{nf(n), f(n)^2\})
\end{aligned}$$

In particular:
If an algorithm runs in polynomial time on a computer, then it can be simulated on a TM in polynomial time. [For $f(n) \in O(n^r)$, we have $nf(n) \in O(n^{r+1})$ and $f(n)^2 \in O(n^{2r})$]

**Exercise 17 (Due 04/28/10)** Can we conclude the following from the observations in this section?

- *If an algorithm runs in exponential time on a computer, then it can be simulated on a TM in exponential time.*
- *If an algorithm runs in linear time on a computer, then it can be simulated on a TM in quadratic time.*

Justify your answer.

# 7 PTime

[[4, Chapter 15.6] and some bits and pieces]

**7.1 Definition**
A language $L$ is *decidable in polynomial time* if there is a standard TM $M$ that accepts $L$ with $tc_M \in O(n^r)$, where $r \in \mathbb{N}$ is independent of $n$. $\mathcal{P}$ (*PTime*) is the *complexity class* of all such languages. [$\mathcal{P}$ is the *set* of all such languages.]

**Exercise 18 (Due 04/28/10)** Show that $\mathcal{P}$ is closed under language complement.

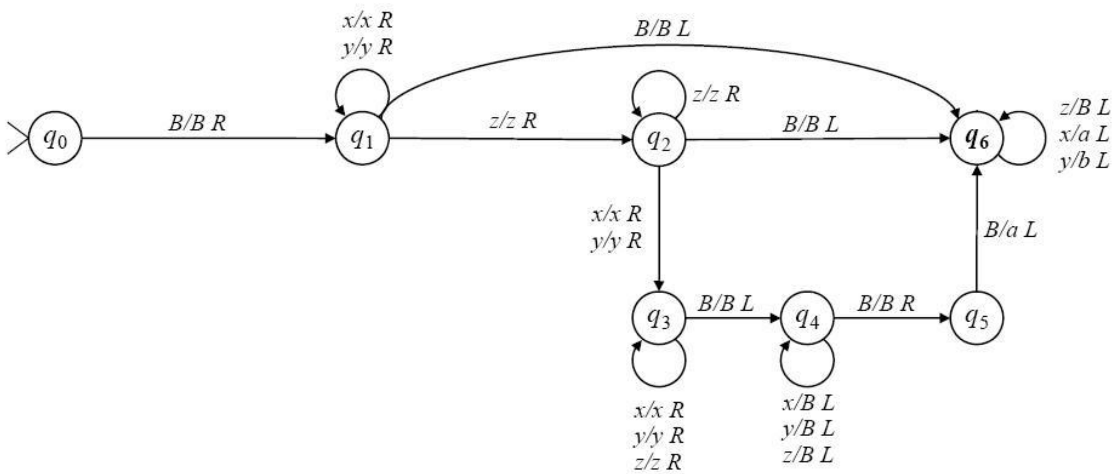
**7.2 The Polynomial Time Church-Turing Thesis**
A decision problem can be solved in polynomial time by using a reasonable sequential model of computation if and only if it can be solved in polynomial time by a Turing Machine.

We have seen that $\mathcal{P}$ is independent of the computation paradigm used:

- standard TMs
- $k$-track TM
- $k$-tape TM
- "realistic" computer

**7.3 Definition**
Let $L$, $Q$, be languages over alphabets $\Sigma_1$ and $\Sigma_2$, respectively. $L$ is *reducible* (*in polynomial time*) to $Q$ if there is a polynomial-time computable function $r : \Sigma_1^* \to \Sigma_2^*$ such that $w \in L$ if and only if $r(w) \in Q$.

**7.4 Example**
The TM




reduces $L = \{x^i y^i z^k \mid i, k \geq 0\}$ to $Q = \{a^i b^i \mid i \geq 0\}$.
Time complexity: $O(n)$.

04/21/10

**Exercise 19 (Due 05/10/10)** Construct a TM which reduces

$$\{a^i b^i a^i \mid i \geq 0\} \text{ to } \{c^i d^i \mid i \geq 0\}.$$

**7.5 Theorem**
Let $L$ be reducible to $Q$ in polynomial time and let $Q \in \mathcal{P}$. Then $L \in \mathcal{P}$.

**Proof:** TM for reduction: $R$.
TM deciding $Q$: $M$.
$R$ on input $w$ generates $r(w)$ as input to $M$.
$\text{length}(r(w)) \leq \max\{\text{length}(w), tc_R(\text{length}(w))\}$.
If $tc_R \in O(n^s)$ and $tc_M \in O(n^t)$, then

$$tc_R(n) + tc_M(\max\{n, tc_R(n)\}) \in O(n^s) + O(\max\{O(n^t), O((n^s)^t)\}) = O(n^{st}).$$

■

**7.6 Definition**
A language (problem) $L$ is

- $\mathcal{P}$*-hard*, if every language in $\mathcal{P}$ is reducible to $L$ in polynomial time.
- $\mathcal{P}$*-complete*, if $L$ is $\mathcal{P}$-hard and $L \in \mathcal{P}$.

**7.7 Remark**
Definition 7.6 (hardness and completeness) are used likewise for other complexity classes. Thereby, reducibility is always considered to be polynomial time.

**7.8 Remark**
In principle, you could use any decision problem for defining a complexity class. E.g., if the *POPI-problem* is to find out, if $n$ potachls fit into a pistochl of size $n$, then a problem/language $L$ is

- *in POPI* if $L$ is reducible to the POPI-problem (in polynomial time),
- *POPI-hard* if the POPI-problem is reducible to $L$ (in polynomial time),
- *POPI-complete* if it is both in POPI and POPI-hard.

Obvious questions:

- Which complexity classes are interesting or useful?
- How do they relate to each other?

In this class, we mainly talk about two complexity classes: $\mathcal{P}$, and $\mathcal{NP}$ (soon).

**Exercise 20 (Due 05/10/10)** By Theorem 5.1, there are problems which are not in $\mathcal{P}$. Go to

`http://qwiki.stanford.edu/wiki/Complexity_Zoo`

and do the following.

1. Find the names of 4 complexity classes which contain $\mathcal{P}$.
2. Find a $\mathcal{P}$-hard problem and describe it briefly in general, but understandable, terms. (You may have to use other sources for background understanding.)

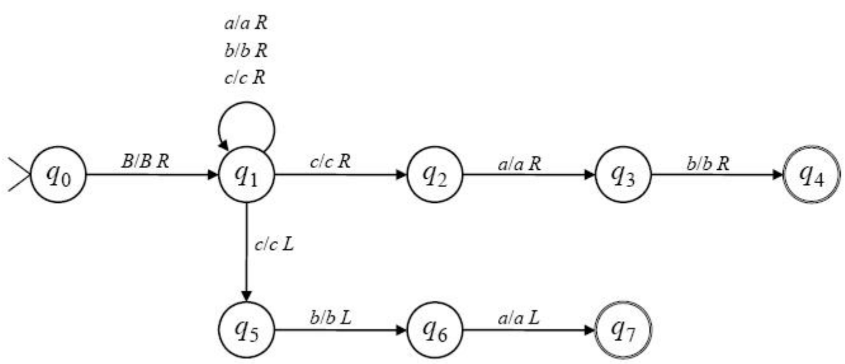
# 8 Nondeterministic Turing Machines and Time Complexity

[4, Chapters 15.1, 15.2 cont., some of Chapter 7.1, and recap Chapter 8.7]

- A nondeterministic (ND) TM may specify any finite number of transitions for a given configuration.
- Transitions are defined by a function from $Q \times \Gamma$ to subsets of $Q \times \Gamma \times \{L, R\}$.
- A computation arbitrarily chooses one of the possible transitions. Input is accepted if there is at least one computation terminating in an accepting state.

**8.1 Definition**
Time complexity $tc_M : \mathbb{N} \to \mathbb{N}$ of an ND TM $M$ is defined s.t. $tc_M(n)$ is the maximum number of transitions processed by input of length $n$.

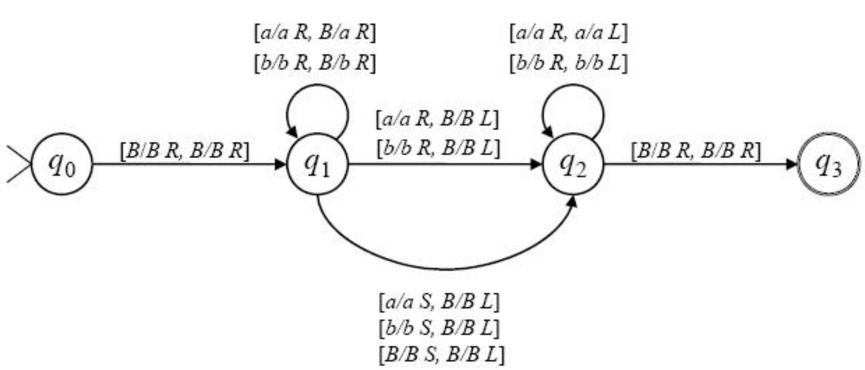
**8.2 Example**
Accept strings with a $c$ preceeded or followed by $ab$.




Complexity: $O(n)$

04/28/10

**8.3 Example**
2-tape ND palindrome finder.




Complexity: $n + 2$ if $n$ is odd, $n + 3$ if $n$ is even.

**Exercise 21** Give a nondeterministic two-tape TM for $\{uu \mid u \in \{a,b\}^*\}$ which is quicker than the TM from Exercise 3.3.

**8.4 Definition**
A language $L$ is accepted *in nondeterministic in polynomial time* if there is an ND TM $M$ that accepts $L$ with $tc_M \in O(n^r)$, where $r \in \mathbb{N}$ is independent of $n$. $\mathcal{NP}$ is the *complexity class* of all such languages.

**8.5 Remark**
$\mathcal{P} \subseteq \mathcal{NP}$. It is currently not known if $\mathcal{P} = \mathcal{NP}$.

**8.6 Theorem**
Let $L$ be accepted by a one-tape ND TM $M$. Then $L$ is accepted by a deterministic TM $M'$ with $tc_{M'}(n) \in O(tc_M(n)c^{tc_M(n)})$, where $c$ is the maximum number of transitions for any state, symbol pair of $M$.

**Proof sketch:** Simulation idea: Use 3-tape TM $M'$. Tape 1 holds input. Tape 2 is used for simulating the tape of $M$. Tape 3 holds sequences $(m_1, \ldots, m_n)$ $(1 \leq m_i \leq c)$, which encode computations of $M$: $m_i$ indicates that, from the (maximally) $c$ choices $M$ has in performing the $i$-th transition, the $m_i$-th choice is selected.
$M$ is simulated as follows:

1. generate a $(m_1, \ldots, m_n)$
2. simulate $M$ according to $(m_1, \ldots, m_n)$
3. if input is not accepted, continue with step 1.

Worst case: $c^{tc_M(n)}$ sequences need to be examined. Simulation of a single computation needs maximally $O(tc_M(n))$ transitions. Thus, $tc_{M'}(n) \in O(tc_M(n)c^{tc_M(n)})$. ■

**Exercise 22** Make a (deterministic) pseudo-code algorithm for an exhaustive search on a tree (i.e., if the sought element is not found, the whole tree should be traversed).

**Exercise 23** Make a non-deterministic pseudo-code algorithm for an exhaustive search on a tree.

**8.7 Definition**
co-$\mathcal{NP} = \{\overline{L} \mid L \in \mathcal{NP}\}$ and co-$\mathcal{P} = \{\overline{L} \mid L \in \mathcal{P}\}$, where $\overline{L}$ denotes the complement of $L$. It is currently not known if $\mathcal{NP} =$ co-$\mathcal{NP}$.

**8.8 Theorem**
If $\mathcal{NP} \neq$ co-$\mathcal{NP}$, then $\mathcal{P} \neq \mathcal{NP}$.

**Proof:** Proof by contraposition:
If $\mathcal{P} = \mathcal{NP}$, then by Exercise 18 we have

$$\mathcal{NP} = \mathcal{P} = \text{co-}\mathcal{P} = \text{co-}\mathcal{NP}.$$

■

**8.9 Theorem**
If there is an $\mathcal{NP}$-complete language $L$ with $\overline{L} \in \mathcal{NP}$, then $\mathcal{NP} = \text{co-}\mathcal{NP}$.

**Proof:** Assume $L$ is a language as stated.
Let $Q \in \mathcal{NP}$. Then $Q$ is reducible to $L$ in polynomial time. This reduction is also a reduction of $\overline{Q}$ to $\overline{L}$.
Combining the TM which performs the reduction with the TM which accepts $\overline{L}$ results in an ND TM that accepts $\overline{Q}$ in polynomial time. Thus $Q \in \text{co-}\mathcal{NP}$.
This shows $\text{co-}\mathcal{NP} \subseteq \mathcal{NP}$. The inclusion $\mathcal{NP} \subseteq \text{co-}\mathcal{NP}$ follows by symmetry. ■

05/10/10

**8.10 Theorem**
Let $Q$ be an $\mathcal{NP}$-complete language. If $Q$ is reducible to $L$ in polynomial time, then $L$ is $\mathcal{NP}$-hard.

**Proof:** If $R \in \mathcal{NP}$, then $R$ is reducible in polynomial time to $Q$, which in turn is reducible in polynomial time to $L$. By composition, $R$ is reducible in polynomial time to $L$. ■

**8.11 Remark**
When moving from languages to decision problems, the representation of numbers may make a difference: Conversion from binary to unary representation is in $O(2^n)$.
Thus, if a problem can be solved in polynomial time with unary input representation, it may not be solvable in polynomial time with binary input representation.
Most reasonable representations of a problem differ only polynomially in length, but not so unary number encoding.
Thus, in complexity analysis, numbers are always assumed to be represented in binary.

**8.12 Definition**
A decision problem with a polynomial solution using unary number representation, but no polynomial solution using binary representation, is called *pseudo-polynomial*.

**Exercise 24** Let $L$ be the language of all strings over $\{a, b\}$ that can be divided into two strings (not necessarily the same length) such that (1) both strings have the same number of $b$s and (2) both strings start and end with $a$. E.g., *abbaababaa* is in $L$ because it can be divided into *abba* and *ababaa*, both of which have 2 $b$s and both of which start and end in $a$. The string *bbabba* is not in $L$.
Give a 2-tape, single track ND TM that accepts $L$.

# 9 SAT is $\mathcal{NP}$-Complete

[4, Chapter 15.8]

Let $V$ be a set of *Boolean variables*.

**9.1 Definition**
An *atomic formula* is a Boolean Variable.
(*Well-formed*) *formulas* are defined as follows.

1. All atomic formulas are formulas.
2. For every formula $F$, $\neg F$ is a formula, called the *negation* of $F$.
3. For all formulas $F$ and $G$, also $(F \vee G)$ and $(F \wedge G)$ are formulas, called the *disjunction* and the *conjunction* of $F$ and $G$, respectively.
4. Nothing else is a formula.

**9.2 Definition**
$\mathbb{T} = \{0, 1\}$ – the set of *truth values*: *false*, and *true*, respectively.
An *assignment* is a function $\mathcal{A} : \mathbf{D} \to \mathbb{T}$, where $\mathbf{D}$ is a set of atomic formulas.
Assignments extend to formulas, via the following *truth tables*.

| $\mathcal{A}(F)$ | $\mathcal{A}(G)$ | $\mathcal{A}(F \wedge G)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

| $\mathcal{A}(F)$ | $\mathcal{A}(G)$ | $\mathcal{A}(F \vee G)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| $\mathcal{A}(F)$ | $\mathcal{A}(\neg F)$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

A formula $F$ is called *satisfiable* if there exists an assignment $\mathcal{A}$ with $\mathcal{A}(F) = 1$.

**9.3 Example**
Determining the truth values of formulas using truth tables:

| $\mathcal{A}(B)$ | $\mathcal{A}(F)$ | $\mathcal{A}(I)$ | $\mathcal{A}(B \wedge F)$ | $\mathcal{A}(\neg(B \wedge F))$ | $\mathcal{A}(\neg I)$ | $\mathcal{A}(\neg(B \wedge F) \vee \neg I)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 |

**Exercise 25** Make the truth table for the formula $\neg(I \vee \neg B) \vee \neg F$.

**Exercise 26** Give a formula $F$, containing only the Boolean variables $A$, $B$, and $C$, such that $F$ has the following truth table.

| $\mathcal{A}(A)$ | $\mathcal{A}(B)$ | $\mathcal{A}(C)$ | $\mathcal{A}(F)$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

**9.4 Definition**
Formulas $F$ and $G$ are (*semantically*) *equivalent* (written $F \equiv G$) if for every assignment $\mathcal{A}$, $\mathcal{A}(F) = \mathcal{A}(G)$.

**9.5 Theorem**
The following hold for all formulas $F$, $G$, and $H$.

| | | |
|---|---|---|
| $F \wedge G \equiv G \wedge F$ | $F \vee G \equiv G \vee F$ | Commutativity |
| $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$ | $(F \vee G) \vee H \equiv F \vee (G \vee H)$ | Associativity |
| $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ | $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$ | Distributivity |
| $\neg\neg F \equiv F$ | | Double Negation |
| $\neg(F \wedge G) \equiv \neg F \vee \neg G$ | $\neg(F \vee G) \equiv \neg F \wedge \neg G$ | de Morgan's Laws |

**Proof:** Straightforward using truth tables. ∎

05/12/10

**9.6 Definition**
A *literal* is an atomic formula (a *positive* literal) or the negation of an atomic formula (a *negative* literal). A *clause* is a disjunction of literals.
A formula $F$ is in *conjunctive normal form* (CNF) if it is a conjunction of clauses, i.e., if

$$F = \left( \bigwedge_{i=1}^{n} \left( \bigvee_{j=1}^{m} L_{i,j} \right) \right),$$

where the $L_{i,j}$ are literals.

**9.7 Theorem**
For every formula $F$ there is a formula $F_1 \equiv F$ in CNF.

**Proof:** skipped ∎

**Exercise 27** Transform $\neg((A \vee B) \wedge (C \vee D) \wedge (E \vee F))$ into CNF.

**Exercise 28** Give an informal, but plausible, argument, why a naive algorithm for converting formulas into CNFs is not in $\mathcal{P}$. [Don't use TMs.]

**9.8 Definition**
The *Satisfiability Problem* (*SAT*) is the problem of deciding if a formula in CNF is satisfiable.

**9.9 Theorem (Cook's Theorem)**
SAT is $\mathcal{NP}$-complete.

**Proof:** later ∎

**9.10 Remark**
SAT is sometimes stated without the requirement that the formula is in CNF – this is equivalent.

**9.11 Proposition**
For any formula $F$, there is an equivalent formula which contains only $\wedge$, $\vee$, and literals. (Called a *negation normal form* (*NNF*) of $F$.)

**Proof:** Apply de Morgan's laws exhaustively. ■

**Exercise 29** Give an informal, but plausible, argument that conversion of a formula into NNF is in $\mathcal{P}$.

**9.12 Definition**
Two formulas $F$ and $G$ are *equisatisfiable* if the following holds: $F$ has a model if and only if $G$ has a model.

**9.13 Proposition**
For all formulas $F_i$ ($i = 1,2,3$), $F_1 \vee (F_2 \wedge F_3)$ and $(F_1 \vee E) \wedge (\neg E \vee (F_2 \wedge F_3))$ are equisatisfiable (where $E$ is a propositional variable not occurring in $F_1$, $F_2$, $F_3$).

**Proof:** skipped ■

**Exercise 30** Use the idea from Proposition 9.13 to sketch a polynomial-time algorithm which converts any formula $F$ into an equisatisfiable formula in CNF. [Hint: First convert into NNF.]

**Exercise 31** Give an informal, but plausible, argument, that the problem "*Decide if a formula is satisfiable*" is $\mathcal{NP}$-complete. [Use Cook's Theorem and Exercise 30.]

[Slideset 2: Proof of Cook's Theorem] 05/17/10

# 10 Excursus: Is $\mathcal{P} = \mathcal{NP}$?

[mainly from memory]

[blackboard]

**Exercise 32 (optional)** Show $|\mathbb{R}| = |\{(1,x) : x \in \mathbb{R}\} \cup \{(2,x) : x \in \mathbb{R}\}|$.

**Exercise 33 (optional)** Show, using a diagonalization argument, that the power set of $\mathbb{N}$ is of higher cardinality than $\mathbb{N}$. [Hint: Consider only infinite subsets of $\mathbb{N}$.]

05/19/10

# 11 If $\mathcal{P} \neq \mathcal{NP}$ ...

[A mix, mainly from [1, Chapter 7], with some from [4, Chapter 17] and other sources.]

## 11.1 Problems "between" $\mathcal{P}$ and $\mathcal{NP}$

$\mathcal{NPC}$ consists of all $\mathcal{NP}$-complete languages.
If $\mathcal{P} \subsetneq \mathcal{NP}$, is $\mathcal{NPI} = \mathcal{NP} \setminus (\mathcal{P} \cup \mathcal{NPC}) \neq \emptyset$?

**11.1 Theorem**
Let $B$ be a recursive language such that $B \notin \mathcal{P}$. Then there exists $D \in P$ s.t. $A = D \cap B \notin \mathcal{P}$, $A$ is (polytime) reducible to $B$ but $B$ is not (polytime) reducible to $A$.

**Exercise 34 (skipped)** Assumed $\mathcal{P} \neq \mathcal{NP}$, why does Theorem 11.1 show that $\mathcal{NPI} \neq \emptyset$?

Iteratively reapplying the argument from Exercise 34 yields an infinite collection of distinct complexity classes "between" $\mathcal{P}$ and $\mathcal{NP}$.
Are there any "natural" candidates for problems in $\mathcal{NPI}$? Perhaps the following.

- **Graph isomorphism:** Given two graphs $G = (V, E)$ and $G' = (V, E')$, is there a bijection $f : V \to V$ such that $(u, v) \in E$ iff $(f(u), f(v)) \in E'$?
- **Composite numbers:** Given $k \in \mathbb{N}$, are there $n, m \in \mathbb{N}$ s.t. $k = m \cdot n$?

## 11.2 The Polynomial Hierarchy

**11.2 Definition**
An *oracle TM* (*OTM*) is a standard TM with an additional *oracle tape* with read-write *oracle head*. It has two additional distinguished states: the *oracle consultation state* and the *resume-computation state*. Also, an *oracle function* $g : \Sigma^* \to \Sigma^*$ is given.
Computation is as for a 2-tape TM, except in the oracle state: If $y$ is on the oracle tape (right of the first blank), then it is rewritten to $g(y)$ (with rest blank) in one step, and the state is changed to the resume state.

Let $C$ and $D$ be two complexity classes (sets of languages). Denote by $C^D$ the class of all languages which are accepted by an OTM of complexity $C$, where computation of the oracle function has complexity $D$.

**11.3 Remark**
$\mathcal{P}^{\mathcal{P}} = \mathcal{P}$ [The one-step oracle consultation can be performed using a TM which runs in polynomial time. Note that there can be at most polynomially many such consultations.]

**Exercise 35** Show: $\mathcal{P}^{\mathcal{NP}}$ contains all languages which are (polynomial-time) reducible to a language in $\mathcal{NP}$.

05/24/10

**11.4 Definition**
The *polynomial hierarchy*:

$$\Sigma_0^p = \Pi_0^p = \Delta_0^p = \mathcal{P}$$

and for all $k \geq 0$

$$\Delta_{k+1}^p = \mathcal{P}^{\Sigma_k^p}$$
$$\Sigma_{k+1}^p = \mathcal{NP}^{\Sigma_k^p}$$
$$\Pi_{k+1}^p = \text{co-}\Sigma_{k+1}^p$$

PH is the union of all classes in the polynomial hierarchy.

**Exercise 36** Show $\Sigma_1^p = \mathcal{NP}$.

**11.5 Remark**
$\Pi_1^p = \text{co-}\mathcal{NP}^{\mathcal{P}} = \text{co-}\mathcal{NP}$
$\Delta_2^p = \mathcal{P}^{\Sigma_1^p} = \mathcal{P}^{\mathcal{NP}}$

**11.6 Remark**
$\Sigma_i^p \subseteq \Delta_{i+1}^p \subseteq \Sigma_{i+1}^p$
$\Pi_i^p \subseteq \Delta_{i+1}^p \subseteq \Pi_{i+1}^p$
$\Sigma_i^p = \text{co-}\Pi_i^p$
It is not known if the inclusions are proper. If any $\Sigma_k^p$ equals $\Sigma_{k+1}^p$ or $\Pi_k^p$, then the hierarchy collapses above $k$. In particular, if $\mathcal{P} = \mathcal{NP}$, then $\mathcal{P} = \text{PH}$.

The following problem may be in $\Sigma_2^p = \mathcal{NP}^{\mathcal{NP}}$:
**Maximum equivalent expression:** Given a formula $F$ and $k \in \mathbb{N}$, is there $F_1 \equiv F$ with $k$ or fewer occurrences of literals?
[$\mathcal{NP}$-hardness: because SAT reduces to it.
In $\Sigma_2^p$: Use SAT (non-CNF version) as oracle. The OTM first guesses $F_1$, then consults the oracle.]

**Exercise 37** Spell out in more detail, how SAT reduces to *maximum equivalent expression.*

# 12 Beyond $\mathcal{NP}$

[Mainly from [4, Chapter 17], plus some from [1, Chapter 7] and other sources.]

*Space complexity*: use modified $k$-tape TM (*off-line TM*) with additional read-only *input tape*, and additional write-only *output tape*. The latter is not needed for language recognition tasks.

**12.1 Definition**
The *space complexity* of a TM $M$ is the function $sc_M : \mathbb{N} \to \mathbb{N}$ s.t. $sc_M(n)$ is the maximum number of tape squares read on any work tape by a computation of $M$ when initiated with an input string of length $n$. (For an ND TM, take the maximum over every possible computation as usual.)

**12.2 Example**
3-tape palindrome recognizer $M$ with $sc_M(n) = O(\log_2(n))$.
Idea: Use work tapes to hold numbers (binary encoding). They are used as counters for identifying and comparing the $i$-th element from the left with the $i$-th element from the right, until a mismatch is found (or the palindrome is accepted).

**12.3 Remark**
Palindrome recognition is in the complexity class LOGSPACE.

**12.4 Theorem**
For any TM $M$, $sc_M(n) \leq tc_M(n) + 1$.

**12.5 Definition**
An off-line TM is said to be $s(n)$ *space-bounded* if the maximum number of tape squares used on a work tape with input of length $n$ is at most $\max\{1, s(n)\}$. (This can also be used with non-terminating TMs.)

**12.6 Theorem (Savitch's Theorem)**
$M$ a 2-tape ND TM with space bound $s(n) \geq \log_2(n)$ which accepts $L$. Then $L$ is accepted by a deterministic TM with space bound $O(s(n)^2)$.

**12.7 Corollary**
$\mathcal{P}$-Space $= \mathcal{NP}$-Space

**Proof:** Obviously, $\mathcal{P}$-Space $\subseteq \mathcal{NP}$-Space.
If $L \in \mathcal{NP}$-Space, then it is accepted by an ND TM with polynomial space bound $p(n)$. Then by Savitch's Theorem, $L$ is accpeted by a deterministic TM with space bound $O(p(n)^2)$. ■

**12.8 Definition**
EXPTIME is the complexity class of problems solvable by a (deterministic) TM in $O\left(2^{g(n)}\right)$ time. NEXPTIME is the corresponding ND class. 2-EXPTIME/N2-EXPTIME are defined similarly with time bound $O(2^{2^{p(n)}})$. (Similarly $n$-EXPTIME – the *exponential hierarchy.*) (EXPSPACE is the corresponding space complexity class.)

**Exercise 38** Show, that EXPSPACE=NEXPSPACE.

**12.9 Remark**
It is known that LOGSPACE $\subseteq$ NLOGSPACE $\subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq$ PH $\subseteq \mathcal{P}$-Space $= \mathcal{NP}$-Space $\subseteq$ EXPTIME $\subseteq$ NEXPTIME $\subseteq$ EXPSPACE $\subseteq$ 2-EXPTIME.
Also known:

- LOGSPACE $\subsetneq \mathcal{P}$-Space
- $\mathcal{P} \subsetneq$ EXPTIME
- $\mathcal{NP} \subsetneq$ NEXPTIME
- $\mathcal{P}$-Space $\subsetneq$ EXPSPACE
- If $\mathcal{P} = \mathcal{NP}$, then EXPTIME = NEXPTIME
- If $\mathcal{P} = \mathcal{NP}$, then $\mathcal{P}$ = PH

**12.10 Remark**
The Web Ontology Language OWL-DL (see [2]) is N2-EXPTIME-complete. The description logic $\mathcal{ALC}$ (see [3]) is EXPTIME-complete.

**Exercise 39** Show: If $\mathcal{NPI} = \emptyset$, then $\mathcal{NP} \neq$ EXPTIME.

**12.11 Example**
$\mathcal{P}$-Space-complete problems:

- **Regular expression non-universality:** Given a regular expression $\alpha$ over a finite alphabet $\Sigma$, is the set represented by $\alpha$ different from $\Sigma^*$?
- **Linear space acceptance:** Given a linear space-bounded TM $M$ and a finite string $x$ over its input alphabet, does $M$ accept $x$?

# 13 $\mathcal{NP}$-complete Problems

[Part of [4, Chapter 16]]

[class presentations]

# References


[1] M. R. Garey and D. S. Johnson. *Computers and Intractability*. Freeman, 1979.

[2] P. Hitzler, M. Krötzsch, B. Parsia, P. F. Patel-Schneider, and S. Rudolph, editors. *OWL 2 Web Ontology Language: Primer*. W3C Recommendation 27 October 2009, 2009. Available from `http://www.w3.org/TR/owl2-primer/`.

[3] P. Hitzler, M. Krötzsch, and S. Rudolph. *Foundations of Semantic Web Technologies*. Chapman & Hall/CRC, 2009.

[4] T. A. Sudkamp. *Languages and Machines*. Addison Wesley, 3rd edition, 2006.